7/6

A TREATISE

ON

CONIC SECTIONS

AND THE

APPLICATION OF ALGEBRA TO GEOMETRY.

By J. HYMERS, M.A.

FELLOW AND TUTOR OF ST JOHN'S COLLEGE, CAMBRIDGE.

CAMBRIDGE:
PRINTED BY JOHN W. PARKER, UNIVERSITY PRINTER;
FOR T. STEVENSON, CAMBRIDGE;
RIVINGTONS; AND LONGMAN & Co., LONDON.

M.DCCC.XXXVII.

CONTENTS.

SECTION I.

ON THE METHODS OF DETERMINING THE POSITION OF A POINT IN A PLANE.

SECTION II.

ON THE STRAIGHT LINE.

SECTION III.

SECTION IV.

ON THE CIRCLE.

SECTION V.

SECTION VI.

ON THE PARABOLA.

SECTION VII.

ON THE ELLIPSE.

SECTION VIII.

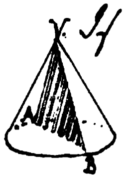
ON THE HYPERBOLA.

SECTION IX.

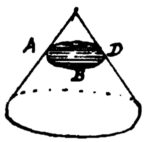
SECTION X.

Conic sections are the figures made by the mutual intersection of a cone and a plane.

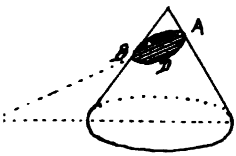
According to the different positions of the cutting plane, there arise 5 diff.^t fig.^s or sections, viz. A triangle, a circle, an ellipse, a parabola, and an hyperbola: the 3 last of wh. are properly called the Conic Sections –



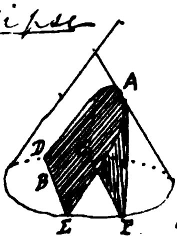
If the cutting plane pass through the vertex of the cone, & any part of the base, the section will be a triangle as VAB.



If the plane cut the cone parallel to the base, or make no angle with it, the section will be a circle. as ABD



When the cone is cut obliquely thro' both sides, or when the plane is inclined to the base in a less angle than the side of the cone is, the section DAB, thus formed is an ellipse



The section ADBE is a parabola when the cone is cut by a plane parallel to the side, or when the cutting plane and the ~~base~~ side of the cone make equal angles with the base.

The section is an hyperbola when the cutting plane makes a greater angle with the base than the side of the cone does, as ACF.

& if the intersection be cont.^d an opp. ~~cone~~ hyp.

CONIC SECTIONS

AND THE

APPLICATION OF ALGEBRA TO GEOMETRY.

SECTION I.

ON THE METHODS OF DETERMINING THE POSITION OF A POINT IN A PLANE.

Rectangular and oblique co-ordinates. Polar co-ordinates.

1. In order to determine the position of a point in a plane, some fixed point in the plane is taken for the origin of co-ordinates; and through it are drawn two fixed lines, called the co-ordinate axes, at right angles to one another.

Then if the perpendicular distances of a point from each of the co-ordinate axes be given, its position will be completely determined.

For let A (fig. 1), be the origin of co-ordinates, $X'AX$, $Y'AY$, the co-ordinate axes, P any point, and PM, PN the perpendiculars let fall from it upon the co-ordinate axes; these perpendiculars are called the rectangular co-ordinates of P, and as their values change for the different points of the plane, they are denoted by the variables x and y.

Then the point P will be determined in position, if we know the values of its two co-ordinates; that is, if we know that for that point $x = a$, $y = b$; for if along AX we

measure $AN = a$, and through N draw an indefinite line parallel to AY, this line will contain all points in the plane whose distance from AY is a, or for which $x = a$, and therefore the point in question; similarly, if we measure along AY the distance $AM = b$, and through M draw an indefinite line parallel to AX, this line will contain the point in question; therefore these two lines MP, NP will by their intersection in P, determine one single position for the point whose co-ordinates are $x = a$, $y = b$; which position, as we see, coincides with the angular point opposite the origin of the rectangle constructed with the sides AN, AM equal to the two given co-ordinates.

2. Instead of the perpendicular PM, its equal AN is commonly used to determine the position of the point P; and the two AN, NP, are called the co-ordinates of P, and are denoted by x and y; the former, for the sake of distinction, being called the abscissa, as being cut off from AX, and the latter, which is parallel to the other axis AY, the ordinate.

When the point is given, and consequently its co-ordinates known, they are usually represented by the first letters of the alphabet a, b, &c. as above; or by the accented letters x', y', or x'', y''; also the axes of the co-ordinates AX, AY, are often called the axis of x and the axis of y.

3. The determination of the point P will not however be complete, unless we take into account the signs of the quantities a, b, in the equations

$$x = a, \quad y = b,$$

in order to measure these distances, when they are positive, along the positive parts AX, AY, of the co-ordinate axes; or along the negative parts AX', AY', of the axes produced in the contrary direction, when they are negative; as is explained in Trigonometry, (Art. 20). For since the co-ordinate axes, which must be supposed to be prolonged indefinitely, form about the origin four angular compartments, there are four positions in which P might be situated, at

absolute distances, a, b, from the co-ordinate axes; and it is only attention to the algebraical signs, with which the values of those distances are affected, that will enable us to select the true one. The direction of the negative abscissæ is quite arbitrary, as is also that of the negative ordinates; we shall however, according to the usual practice, measure the positive abscissæ from the origin towards the right, and the negative abscissæ from the origin towards the left; and the positive ordinates we shall measure upwards from the axis of x, and the negative ordinates, downwards. Hence if the point P be situated in the compartment XAY, both its co-ordinates are positive; if in the opposite compartment $X'AY'$, both are negative; and for points in the compartments $X'AY$, XAY', we must have respectively

$$x = -a, \quad y = b; \quad x = a, \quad y = -b;$$

also for points in the axis of x, and axis of y, we shall have respectively

$$x = a, \quad y = 0; \quad x = 0, \quad y = b;$$

and for the origin, $x = 0$, $y = 0$.

4. Sometimes it is requisite to take the co-ordinate axes not at right angles, but inclined at a given angle to one another; in which case the system of co-ordinates is called oblique. Thus (fig. 2), if XAX', YAY', be two lines drawn through the point A, and intersecting one another at a given angle; and if from any point P in the plane XAY, PM, PN, be drawn respectively parallel to AX, AY, and meeting those axes in M and N; PM, or its equal AN, and NP are the co-ordinates of P, referred to the oblique axes AX, AY.

5. To find the distance of a point from the origin in terms of its co-ordinates.

Let P be the point (fig. 1).

$AN = x'$, $NP = y'$ its given co-ordinates. Join AP, and let $AP = d$; then from the triangle ANP, right-angled at N,

$$AP^2 = AN^2 + NP^2, \text{ or } d^2 = x'^2 + y'^2,$$
$$\therefore \quad d = \sqrt{x'^2 + y'^2}.$$

6. To find the distance between two points in terms of their co-ordinates, and the angle of inclination of the line which joins them to the axis of x.

Let P' be a point (fig. 3), whose co-ordinates are x' and y'; and P any other point whose co-ordinates are x and y; join PP', and draw $P'Q$ parallel to AX meeting the ordinate of P in Q; then from the triangle PQP', right-angled at Q,

$$PP'^2 = P'Q^2 + PQ^2,$$
$$\text{or} \quad d^2 = (x - x')^2 + (y - y')^2,$$
$$\therefore \quad d = \sqrt{(x - x')^2 + (y - y')^2}.$$

Both in this formula and in that of Art. 5, we take the radical with a positive sign, as the question only relates to the absolute distance of the points.

Next let a be the angle which PP' forms with $P'Q$, and which is equal to the angle at which, if produced, PP' would be inclined to the axis of x,

$$\text{then} \quad \tan a = \frac{PQ}{P'Q} = \frac{y - y'}{x - x'}.$$

7. Suppose the co-ordinates to be oblique, and the axes of the co-ordinates to be inclined to one another at an angle ω; then, for the distance of a point from the origin, we have (fig. 2)

$$AP^2 = AN^2 + NP^2 - 2AN.NP \cos ANP,$$
$$\text{but} \quad \cos ANP = -\cos XAY = -\cos \omega,$$
$$\therefore \quad d^2 = x^2 + y^2 + 2xy \cos \omega;$$

and for the distance between two points we have, in a similar manner, from the triangle PQP' (fig. 3), in which the angle $PQP' = \pi - \omega$,

$$d^2 = (x - x')^2 + (y - y')^2 + 2(x - x')(y - y') \cos \omega.$$

8. There is also another mode of determining the position of a point in a plane, viz. by means of its distance from a given point or pole, and the angle which that distance makes with a fixed line or axis in the plane.

Let A (fig. 4), be the origin or pole, and AX a fixed line or axis; and P any point in the plane passing through AX. Join AP, then AP is called the radius vector, and is usually denoted by r, and the angle PAX is called the angle of revolution, and is denoted by θ; and r and θ are called the polar co-ordinates of P; and if given values $r = d$, $\theta = a$, be assigned for them, the position of P will be completely determined.

9. To express the distance of two points from one another in terms of their polar co-ordinates.

Let P' be a point (fig. 4) whose polar co-ordinates are r' and θ', and P any other point whose co-ordinates are r and θ; then the angle $PAP' = \theta - \theta'$, and joining PP', we get from the triangle PAP',

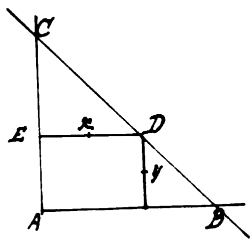
$$PP' \text{ or } d = \sqrt{r^2 + r'^2 - 2rr' \cos(\theta - \theta')}.$$

Equation to a curve. Locus of an equation.

10. Knowing in this way how to determine the position of a point in a plane by means of its co-ordinates, conceive a curve line to be traced on a plane, and each of its points to be referred to two known axes; and that we have between the abscissa and ordinate of each point an invariable relation. In a great many cases it happens that this relation is of a nature to be expressed by an equation between the abscissa and ordinate; and this equation, when obtained, enables us to find either of these quantities by means of the other; so that giving to the abscissa, for instance, arbitrary values, we can deduce from the equation corresponding values of the ordinate; and we thus determine as many points of the curve as we please.

The equation which expresses generally the invariable relation of the abscissa and ordinate of every point of a curve to one another, is called the equation to the curve: and, conversely, the curve is called the locus of the equation. Similarly, the equation which expresses the invariable relation of the radius vector and angle of revolution of a curve, is called the polar equation to the curve.

11. All lines are regular or irregular; irregular lines, described, as it is termed, liberâ manu, are not subjects of mathematical investigation, and cannot be represented by equations; but regular lines which are described according to some constant law, which determines the position of all their points, can be represented by equations. This idea of regular lines agrees with the geometrical loci of the ancients. They gave that name to those lines of which every point was equally proper to solve an indeterminate geometrical problem. Thus a circle was said to be the locus of the vertices of all triangles on a given base and having a given vertical angle. Des Cartes first adopted the method of expressing by an algebraical equation the nature of lines. The object of the following sections will be to investigate the equations to curves, and from those equations to discover their geometrical properties, by means of interpretations made according to the laws of Algebra.



13. Let $AB = a$: $AC = b$: $DE = x$

then $a - x : y :: a : b$

product of extremes = product of means.

$$\therefore b(a-x) = ay$$

$$y = \frac{b(a-x)}{a}$$

SECTION II.

ON THE STRAIGHT LINE.

Straight line referred to rectangular co-ordinates.

12. We will now suppose the locus of the point P to be a straight line, as defined in Geometry; and proceed to investigate its equation by means of some of its properties; that is, an equation expressing an invariable relation, which is satisfied by the co-ordinates of every point in the line.

13. To find the equation to a straight line.

Let A be the origin (fig. 5), AX the axis of x, AY that of y, RT the given straight line, P any point in it, and $AN = x$, $PN = y$ the co-ordinates of the point P.

Let $AB = c$, and the tangent of angle $PTN = m$. Draw BQ parallel to AX meeting PN in Q; then $PQ = BQ \cdot \tan PBQ = AN \cdot \tan PTN = mx$,

$$\text{and } PN = PQ + QN = PQ + AB,$$

$$\therefore\ y = mx + c;$$

and as this relation is satisfied by the co-ordinates of every point in the line, it is the equation required.

14. The meanings of the constants m and c are to be particularly observed; c is the part of the axis of y intercepted between the straight line and the origin, or the ordinate through the origin; m is the tangent of the angle which that part of the line which falls above the axis of x makes with the axis of x produced in the positive direction. They remain the same for the same line, but are different

for different lines, and are called **arbitrary constants**; every straight line has two of them, and therefore a straight line may be drawn fulfilling two conditions.

The equation $y = mx + c$, which is the most convenient form and the one commonly employed, represents a straight line when determined by the conditions of passing through a known point in the axis of y, and making a given angle with a fixed line, viz. the axis of x; so that m is a number or ratio, denoting the tangent of the angle; and c denotes a line, viz. the distance from the origin of the point in the axis of y through which the line passes.

15. The equation to a straight line may also be put under the two following forms, which are sometimes useful.

Let $AT = -a$, the negative sign being prefixed because AT is measured towards the left from A; then

$$m = \tan BTA = -\frac{c}{a};$$

$$\therefore\ y = -\frac{c}{a}x + c,$$

$$\text{or } \frac{x}{a} + \frac{y}{c} = 1,$$

the equation to a straight line when determined by means of the portions of the co-ordinate axes intercepted between it and the origin.

Also if a perpendicular upon the straight line from the origin, $AD = p$, and the angle which the perpendicular forms with the axis of x produced in the positive direction, $DAX = \alpha$, we have $\tan BTA = -\cot\alpha$ and $c = \dfrac{p}{\sin\alpha}$, therefore

$$y = -x\cot\alpha + \frac{p}{\sin\alpha};\quad \text{or } y\sin\alpha + x\cos\alpha = p,$$

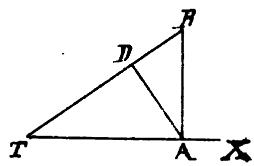
the equation to a straight line when determined by the perpendicular upon it from the origin.

15. $y = -\frac{c}{a}x + c$

$ya = ca - cx$

$\frac{ya}{c} + x = a$

$\frac{y}{c} + \frac{x}{a} = 1$



$\tan BTA = \cot DAT = -\cot\alpha.$

ΔDAT is sim.r to ΔDAB

$\therefore DBA = \alpha$

$\therefore \sin\alpha = \frac{p}{c}$

$\therefore c = \frac{p}{\sin\alpha}$

16. The indeterminate equation of the first degree between two variables, is in its most general form

$$Ax + By + C = 0,$$

which in all cases is the equation to a straight line. For by putting

$$\frac{A}{B} = -m, \quad \frac{C}{B} = -c,$$

we reduce it to the form

$$y - mx - c = 0, \text{ or } y = mx + c,$$

which coincides with the equation to a straight line.

17. A straight line may always be determined from its equation

$$y = mx + c,$$

when the constants m and c are known.

First consider the equation $y = mx$, which represents a line passing through the origin; assume for x any positive value $AN = x'$ (fig. 6), take for y the value $NP = mx'$ (supposing m a positive quantity) and join AP, this is the required line. But if m be negative, so that the equation is $y = -mx$, taking $AN = x'$ and NP measured downwards $= mx'$ (fig 7), and joining AP, we have the line required.

We can now readily construct any line whatever whose equation is

$$y = mx + c.$$

For if in this equation, we give to x the same values that we assigned to it in the equation $y = mx$, the difference of the corresponding values of y will be constant and equal to c; the straight line which is the locus of $y = mx + c$ is consequently parallel to the line AP (fig. 6) determined by $y = mx$; if therefore we take AB or AB' equal to c, according as c is affected with a positive or negative sign, and draw BD, or $B'D'$ parallel to AP, we shall have the straight line required. If m be negative, so that the pro-

posed equation is $y = -mx + c$, then we must take AB or AB' (fig. 7) equal to c, and draw BD or $B'D'$ parallel to AP.

18. As the straight line is determined when any two points are known through which it passes, the position of the line which is the locus of any indeterminate equation of the first degree, may also be assigned by determining two of its points; and for this purpose the points most convenient are those in which it cuts the axes of x and y; the co-ordinates of which are obtained by making x and y successively zero in the given equation.

Thus if the equation be $y = mx + c$, in which m and c are positive, making $x = 0$, we have $y = AB = c$ (fig. 5), and making $y = 0$ we have $x = AT = -\frac{c}{m}$. Hence joining TB and producing it indefinitely, we have the line required.

The distance of the points of intersection from the origin, determined in this manner, must of course be measured along the positive or negative parts of the co-ordinate axes, accordingly as they are affected with positive or negative signs.

Problems relative to the straight line.

19. These principles being laid down, we proceed to the resolution of several problems relative to the straight line, the results of which are of great use; as its equation contains two disposable constants, they may be determined so as to make the line fulfil various conditions; as, for instance, to pass through two given points, to pass through a given point and be parallel, or perpendicular, or inclined at a known angle, to a given line, and so on.

20. To find the equation to a straight line which shall pass through two given points, whose co-ordinates are x', y', x'', y''.

$$y - y' = \frac{y'' - y'}{x'' - x'}(x - x')$$

$$\begin{aligned}
y &= \frac{(y'' - y')(x - x')}{x'' - x'} + y' \\
&= \frac{y''x - y'x - y''x' + y'x' + y'x'' - y'x'}{x'' - x'} \\
&= \frac{y'' - y'}{x'' - x'}x + \frac{y'x'' - y''x'}{x'' - x'}
\end{aligned}$$

Any point in the line, of which the co-ordinates are x and y, being assumed, we have $y = mx + c$. But since x' and y' are also co-ordinates of a point in the same line, they will satisfy this equation,

$$\therefore\ y' = mx' + c.$$

Hence subtracting this equation from the former, we have

$$y - y' = m(x - x').$$

This is the equation to a line fulfilling one condition, viz. passing through the point (x', y'), and since m is arbitrary, an infinite number of lines may be so drawn. But since the line is moreover to pass through the point (x'', y''),

$$\therefore\ y'' - y' = m(x'' - x'), \text{ or } m = \frac{y'' - y'}{x'' - x'};$$

hence substituting this value of m in the above equation, we get

$$y - y' = \frac{y'' - y'}{x'' - x'}(x - x'),$$

the equation required, which also may be written

$$y = \frac{y'' - y'}{x'' - x'}x + \frac{y'x'' - x'y''}{x'' - x'},$$

which is of the general form $y = mx + c$.

21. To find the equation to a straight line which shall pass through a given point and be parallel to a given straight line.

Let the equation to the given line be

$$y = mx + c,$$

where m and c are known; and the equation to the required line

$$y = m'x + c',$$

where m' and c' are unknown; then in order that these lines may be parallel we must have $m = m'$, for they are

respectively parallel to lines passing through the origin whose equations are

$$y = mx, \quad y = m'x,$$

and these two latter lines must coincide, since the two former are parallel; consequently $m = m'$. The equation to the line parallel to the given line then becomes

$$y = mx + c',$$

where c' remains indeterminate, since there is an infinite number of lines which are parallel to a given line; but if it be required to pass through a given point (x', y') we must have

$$y' = mx' + c', \text{ which gives } c'.$$

Subtracting therefore this from the preceding, c' will disappear, and we have for the equation to the required parallel

$$y - y' = m(x - x').$$

22. Having given the equations to two straight lines, to determine their point of intersection.

Let the equations to the two lines be

$$y = mx + c.$$
$$y = m'x + c'.$$

At the point where these lines intersect, they have the same co-ordinates, and conversely their co-ordinates are not equal at any other point except that in which they intersect; hence for that point only we have

$$mx + c = m'x + c',$$

which gives

$$x = \frac{c - c'}{m' - m},$$

and substituting this value for x in the equation to one of the lines, we get

$$y = \frac{m'c - mc'}{m' - m}.$$

$\times \quad y = mx + c$

$$= m\left(\frac{c-c'}{m'-m}\right) + c$$

$$= \frac{mc - mc' + m'c - mc}{m'-m}$$

$$= \frac{m'c - mc'}{m'-m}$$

$$* \quad \sin\varphi = \frac{\tan\varphi}{\sqrt{1+\tan^2\varphi}}$$

$$= \frac{\frac{m-m'}{1+mm'}}{\sqrt{1+\left(\frac{m-m'}{1+mm'}\right)^2}}$$

$$= \frac{\frac{m-m'}{1+mm'}}{\sqrt{\frac{1+2mm'+m^2m'^2+m^2-2mm'+m'^2}{(1+mm')^2}}}$$

$$= \frac{m-m'}{\sqrt{1+m^2m'^2+m^2+m'^2}}$$

$$= \frac{m-m'}{\sqrt{1+m^2}\cdot\sqrt{1+m'^2}}$$

$$\dagger \quad \cos\varphi = \frac{1}{\sqrt{1+\tan^2\varphi}}$$

$$= \frac{1}{\sqrt{\frac{1+2mm'+m^2m'^2+m^2-2mm'+m'^2}{(1+mm')^2}}}$$

$$= \frac{1+mm'}{\sqrt{1+m^2}\cdot\sqrt{1+m'^2}}$$

When $m' = m$ these values become infinite, as ought to happen, for the lines are then parallel; when $c' = c$ as well as $m' = m$, the values become $\frac{0}{0}$, that is to say, indeterminate, which likewise ought to happen, as the lines then become coincident in all their points.

23. Having given the equations to two lines, to find the angle between them.

$$\text{Let } y = mx + c,$$
$$y = m'x + c',$$

be the equations to the lines BT, $B'T'$, (fig. 8); and a, a', the angles at which they are respectively inclined to the axis of x, so that $\tan a = m$, $\tan a' = m'$; and let ϕ be the angle between them.

Then angle $BPB' = \angle PTA - \angle PT'A$,

$$\text{or } \phi = a - a'; \ \therefore \ \tan\phi = \frac{\tan a - \tan a'}{1 + \tan a . \tan a'},$$

$$\text{or } \tan\phi = \frac{m - m'}{1 + mm'}.$$

Similarly, $\sin\phi = \sin a . \cos a' - \cos a . \sin a' = \dfrac{m - m'}{\sqrt{1+m^2} . \sqrt{1+m'^2}}$,

$$\cos\phi = \frac{1 + mm'}{\sqrt{1 + m^2} . \sqrt{1 + m'^2}}.$$

24. Hence, in order that the two lines may be parallel, we must have $\tan\phi = 0$, or $m - m' = 0$, as before.

And in order that they may be at right angles to one another, we must have $\tan\phi = \infty$, or $1 + mm' = 0$.

25. To find the equation to a straight line which passes through a given point, and makes a given angle with a given straight line.

Let BT (fig. 8) be the given line, and $y = mx + c$ its equation, therefore $\tan PTA = m$.

$B'T'$ the required line whose equation may be assumed to be $y - y' = m'(x - x')$, since it passes through the point (x', y'), then $\tan PT'A = m'$, also let $\tan TPT' = t$ a given quantity; then because

$$\angle PT'A = PTA - TPT', \text{ or } a' = a - \phi,$$

$$\tan a' = \frac{\tan a - \tan \phi}{1 + \tan a . \tan \phi}, \text{ or } m' = \frac{m - t}{1 + mt};$$

therefore $y - y' = \dfrac{m - t}{1 + mt}(x - x')$ is the required equation.

26. If the lines are to be parallel, $t = 0$, and as before the equation is

$$y - y' = m(x - x').$$

If the lines are to be perpendicular to one another $t = \infty$, and therefore the equation is

$$y - y' = -\frac{1}{m}(x - x'). \quad *$$

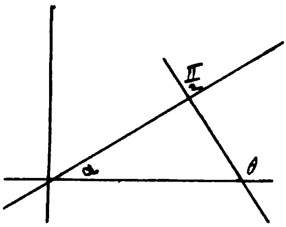
Hence if the coefficient of x in one equation be the reciprocal of the coefficient of x in the other with a contrary sign, that is, if the equations be $y = mx + c$, $y = -\dfrac{1}{m}x + c'$, the lines which they represent are at right angles to one another; and in that case the co-ordinates of their points of intersection are (Art. 22),

$$x = \frac{m(c' - c)}{1 + m^2}, \quad y = \frac{c + m^2 c'}{1 + m^2}.$$

27. Having given the co-ordinates of a point and the equation to a straight line; to find the length of the perpendicular dropped from the point upon the line.

Let x', y', be the co-ordinates of the given point, and $y = mx + c$ the equation to the given line, then the equation to the perpendicular will be

$$y - y' = -\frac{1}{m}(x - x').$$



26. To find the equation to a st. line perpendicular to a line whose =n is $\frac{x}{a}+\frac{y}{b}=1$

* $t=\infty$ because $\tan 90^\circ=\infty$

$$\therefore y-y'=\frac{m-\infty}{1+m\infty}(x-x')=-\frac{\infty}{m\infty}(x-x')=-\frac{1}{m}(x-x')$$

In order to get the co-ordinates of the point of intersection of the given line and perpendicular, we must deduce from their equations values of x and y; to make the elimination more easy, put the first $y = mx + c$ under the form

$$y - y' = m(x - x') + c + mx' - y';$$

combining this with the equation to the perpendicular, and taking for the unknown quantities, the differences $y - y'$, $x - x'$, we get

$$x - x' = \frac{m(y' - mx' - c)}{1 + m^2}, \quad y - y' = -\frac{y' - mx' - c}{1 + m^2},$$

values from which it is easy to deduce the co-ordinates x and y of the foot of the perpendicular. But if we denote by p the length of the perpendicular intercepted between the point and the given line, we have

$$p = \sqrt{(x - x')^2 + (y - y')^2};$$

therefore, putting for $x - x'$ and $y - y'$ their values,

$$p = \pm \frac{y' - mx' - c}{\sqrt{1 + m^2}}.$$

As the value of p must be positive, we must take the upper or lower sign, according as the numerator $y' - mx' - c$ is positive or negative.

If the given point is situated in the origin, $x' = 0$, $y' = 0$, and the value of p is reduced to

$$p = \pm \frac{c}{\sqrt{1 + m^2}}.$$

28. The result of the preceding Article may be readily obtained as follows.

Let $y = mx + c$ be the equation to the given line MT (fig. 9),

$AN = x'$, $PN = y'$ the co-ordinates of the point P,

then the perpendicular $PQ = PR \cos RPQ = PR \cos RTN$.

But $PN = y'$, and $RN = mx' + c$;

$$\therefore\ PR = y' - mx' - c;$$

$$\text{also } \cos RTN = \frac{1}{\sqrt{1 + \tan^2 RTN}} = \frac{1}{\sqrt{1 + m^2}};$$

$$\therefore \text{ the perpendicular } p = \frac{y' - mx' - c}{\sqrt{1 + m^2}}.$$

29. Hence also, if through a given point a line be drawn cutting a given line at a known angle, we can find the distance of the given point from the point of intersection of the lines. For if PS be a line passing through the point P and cutting the line BM (fig. 9) at an angle $PSR = \alpha$, drawing PQ perpendicular to BM, we have

$$SP \sin \alpha = QP = \frac{y' - mx' - c}{\sqrt{1 + m^2}};$$

$$\therefore\ SP = \frac{y' - mx' - c}{\sin \alpha \sqrt{1 + m^2}}.$$

Straight Line referred to Oblique Co-ordinates.

30. To find the equation to a straight line referred to oblique co-ordinates.

Let the axes of the co-ordinates be inclined to one another at an angle ω, and suppose the line PT to cut the axis of x at an angle $= \alpha$. Let $AN = x$, $NP = y$, be co-ordinates of any point P, and draw BQ parallel to AX (fig. 10), meeting the ordinate of P in Q;

$$\text{then } \frac{PQ}{BQ} = \frac{\sin PBQ}{\sin BPQ} = \frac{\sin \alpha}{\sin(\omega - \alpha)}, \quad \text{or } PQ = \frac{x \sin \alpha}{\sin(\omega - \alpha)},$$

$$\text{and } NQ = AB = c;$$

$$\therefore\ y = \frac{x \sin \alpha}{\sin(\omega - \alpha)} + c,$$

the required equation.

Hence if a straight line referred to oblique axes, be represented by the equation

$$y = mx + c,$$

m, the coefficient of x, expresses the ratio of the sines of the angles which the line makes respectively with the axes of x and y; and c, as before, is the ordinate through the origin.

In using the equation

$$y = \frac{x \sin \alpha}{\sin (\omega - \alpha)} + c,$$

we must remember, with respect to the constants involved, (1) that c is a positive or negative quantity, according as the line cuts the axis of y above or below the origin; (2) that ω is the angle YAX formed by the positive parts of the co-ordinate axes, and not the adjacent angle YAX'; and (3) that α is the angle PTX formed by the portion of the line which is situated above the axis of x, with the positive part of that axis.

31. It is easily seen that when a line is determined by the portions of the co-ordinate axes intercepted between it and the origin, its equation is of precisely the same form as when the co-ordinates are rectangular; for let $AB = c$, $AT = -a$, and x and y be the co-ordinates of any point P; we get from the similar triangles ABT, PBQ

$$\frac{c}{-a} = \frac{y - c}{x}, \quad \text{or } \frac{x}{a} + \frac{y}{c} = 1.$$

Also when a line is determined by the condition of passing through two given points, or of being parallel to a given line, its equation is of the same form whether the co-ordinates be rectangular or oblique; in the following cases the results are different.

32. To find the angle between two lines whose equations are given, referred to oblique axes.

First, to calculate the angle which a line whose equation is given, makes with the axis of x. Let $y = mx + c$ be its equation, and ω the angle of inclination of the axes, and let the line make an angle a with the axis of x, and therefore an angle $\omega - a$ with the axis of y, then

$$\frac{\sin a}{\sin(\omega - a)} = m,$$

$$\text{which gives } \tan a = \frac{m \sin \omega}{1 + m \cos \omega}.$$

Next, let $y = m'x + c'$ be the equation to another line referred to the same axes, making an angle a' with the axis of x;

$$\therefore \tan a' = \frac{m' \sin \omega}{1 + m' \cos \omega}.$$

Let ϕ be the angle between the lines,

$$\text{then } \tan \phi = \tan(a' - a) = \frac{\tan a' - \tan a}{1 + \tan a' . \tan a}$$

$$= \frac{(m' - m) \sin \omega}{1 + mm' + (m + m') \cos \omega}.$$

33. Hence if $\phi = 90^0$, then

$$1 + mm' + (m + m') \cos \omega = 0;$$

$$\therefore m' = -\frac{1 + m \cos \omega}{m + \cos \omega},$$

the condition in order that two lines referred to oblique axes, may be perpendicular to one another.

34. To find the perpendicular distance of a given point from a given line referred to oblique axes.

Let x', y', be the co-ordinates of the given point Q (fig. 11), $y = mx + c$ the equation to the given line CN which makes an angle a with the axis of x; and let QP

be the perpendicular let fall from Q upon CN; then $QP = QN \,.\, \sin(\omega - \alpha)$

$$= \frac{y' - mx' - c}{m} \sin \alpha = \frac{(y' - mx' - c)\sin\omega}{\sqrt{1 + 2m\cos\omega + m^2}},$$

$$\text{since } \tan\alpha = \frac{m\sin\omega}{1 + m\cos\omega}.$$

Straight Line referred to Polar Co-ordinates.

35. To find the polar equation to a straight line.

Let A be the pole (fig. 12), XA the initial line PT the proposed straight line, and $AP = r$, $XAP = \theta$ the polar co-ordinates of any point P in it; AQ a perpendicular upon it from the pole equal p, and XAQ the angle which that perpendicular makes with the initial line equal α,

$$\text{then } AP = AQ \sec QAP,$$

$$\text{or } \quad r = p\sec(\theta - \alpha).$$

If the proposed line be perpendicular to the initial line so that $\alpha = 0$, the equation becomes

$$r = p\sec\theta.$$

36. Since the polar equation to a straight line

$$r = p\sec(\theta - \alpha)$$

$$\text{becomes} \quad r\cos(\theta - \alpha) = p,$$

$$\text{or} \quad r\cos\theta \,.\, \cos\alpha + r\sin\theta \,.\, \sin\alpha = p,$$

it appears that every equation of the form

$$Ar\cos\theta + Br\sin\theta + C = 0,$$

is the polar equation to a straight line.

---

37. The following Problems are added to illustrate the use of some of the preceding results.

(1) To find the equation to a straight line which shall pass through the point of intersection of two given lines, and bisect the angle between them.

Let CB, CB' (fig. 13) be the two given lines, determined by the equations $y = mx + c$, $y = m'x + c'$, and making the angles α, α', with the axis of x; CE the required line making an angle θ with that axis;

$$\text{then } \theta = \alpha' + \frac{\alpha - \alpha'}{2} = \frac{\alpha + \alpha'}{2};$$

$$\therefore \cot\theta = \cot\frac{\alpha + \alpha'}{2} = \frac{1}{\sin(\alpha + \alpha')} + \frac{1}{\tan(\alpha + \alpha')}$$

$$= \frac{\sqrt{1 + m^2}.\sqrt{1 + m'^2}}{m + m'} + \frac{1 - mm'}{m + m'};$$

$$\therefore y - \frac{m'c - mc'}{m' - m} = \frac{m + m'}{\sqrt{1 + m^2}.\sqrt{1 + m'^2} + 1 - mm'}.\left(x - \frac{c - c'}{m' - m}\right),$$

(Art. 22), the required equation.

(2) To find the polar equation to a line which passes through two points whose polar co-ordinates are given.

Let P', P'' (fig. 14) be the two given points determined by the polar co-ordinates r', α', r'', α'' respectively, and P any other point whose co-ordinates are r and θ;

$$\text{then } 1 = \frac{PP'' - PP'}{P'P''} = \frac{PP''}{P'P''} - \frac{PP'}{P'P''};$$

$$\text{but } \frac{PP''}{P'P''} = \frac{AP \sin PAP''}{AP' \sin P'AP''} = \frac{r \sin(\theta - \alpha'')}{r' \sin(\alpha' - \alpha'')}.$$

$$\text{Similarly } \frac{PP'}{P'P''} = \frac{r \sin(\theta - \alpha')}{r'' \sin(\alpha' - \alpha'')};$$

$$\therefore \frac{r \sin(\theta - \alpha'')}{r' \sin(\alpha' - \alpha'')} - \frac{r \sin(\theta - \alpha')}{r'' \sin(\alpha' - \alpha'')} = 1.$$

SECTION III.

ON THE TRANSFORMATION OF CO-ORDINATES.

---

38. As the equation to a curve does not remain the same when we change the axes of the co-ordinates to which it is referred; it is of great importance in investigating the form and properties of a curve from its equation, to give the origin of the co-ordinates such a position, and the axes such directions and inclination, as will allow the equation to the curve to appear under the simplest form possible.

Moreover it is frequently required, when we know the equation to a curve referred to an assumed system of axes, to find the equation which represents the same curve when referred to new axes, whose positions are given with respect to the former.

This will be effected, when we know for any point, the values of the old co-ordinates in terms of the new ones; for then, by substituting these values in the proposed equation, we shall obtain a relation between the new co-ordinates, which is true for every point of the curve under consideration. In this consists the transformation of co-ordinates. The problem therefore to be resolved is, to express the primitive co-ordinates of any point in terms of the new co-ordinates, which we shall now consider in the following separate cases.

39. To change the origin of co-ordinates without altering the direction of the axes.

Let A (fig. 15) be the origin of the co-ordinates, and $AN = x$, $PN = y$, the co-ordinates of any point P; $AM = h$,

$MA' = k$, the co-ordinates of the new origin A', $A'N' = x'$, $N'P = y'$, the co-ordinates of the same point P referred to the new axes $A'X'$, $A'Y'$, parallel to the former; then,

$$x = AM + A'N' = h + x' \text{ and } y = A'M + N'P = k + y',$$

and if these values be substituted for x and y, we shall have the equation to the curve referred to the origin A', the axes retaining their former directions.

40. To change the directions of the axes without altering the origin, supposing both systems to be rectangular.

Let $XAX' = YAY' = \theta$ (fig. 16) be the angle at which the new axes AX', AY' are inclined to the original axes AX, AY.

Let $AN = x$, $PN = y$, $AN' = x'$, $PN' = y'$, be the co-ordinates of the same point P referred to these respective axes.

Draw $N'Q$, $N'M$ parallel and perpendicular to AX,

$$\text{then } x = AM - QN' = x' \cos\theta - y' \sin\theta,$$

$$y = MN' + PQ = x' \sin\theta + y' \cos\theta,$$

and if these values be substituted for x and y, we shall have the equation to the curve referred to the axes AX', AY'.

41. To change the direction of the axes without altering the origin, supposing both systems to be oblique.

Let AX, AY (fig. 17) be the primitive axes inclined to one another at an angle $XAY = \omega$, and AX', AY' the new axes determined by the angles $X'AX = \alpha$, $Y'AX = \beta$, which they respectively form with the primitive axis of x.

Let P be any point, $AN = x$, $NP = y$ its primitive co-ordinates, and $AN' = x'$, $N'P = y'$ its new co-ordinates. Then denoting by xy the angle XAY contained by the

axes of x and y produced in the positive directions, and similarly of the others, we have

$$x \sin xy = x' \sin x'y + y' \sin y'y,$$

each of these expressing the perpendicular distance of P from AY;

$$\therefore\ x = x' \frac{\sin(\omega - \alpha)}{\sin \omega} + y' \frac{\sin(\omega - \beta)}{\sin \omega}.$$

Also, $y \sin xy = x' \sin x'x + y \sin y'x,$

each of these expressing the perpendicular distance of P from AX;

$$\therefore\ y = x' \frac{\sin \alpha}{\sin \omega} + y' \frac{\sin \beta}{\sin \omega}.$$

In making use of these formulæ, we must recollect that ω represents the angle XAY contained by those portions of the primitive axes along which the positive co-ordinates are measured, and varies from zero to π; and that α and β denote the angles formed by AX', AY', the positive directions of the new axes, with the positive part AX of the primitive axis of x; and that each of these may receive any value from zero to 2π.

42. These formulæ are not often employed in the above general state; and we may deduce from them, as particular cases, the following, the use of which is more frequent.

First, by making $\omega = \frac{\pi}{2}$ and $\beta = \frac{\pi}{2} + \alpha$, we fall upon the formulæ of the second case for passing from one system of rectangular co-ordinates to another system also rectangular.

Secondly, by making $\omega = \frac{\pi}{2}$ we obtain the formulæ for passing from a rectangular to an oblique system of co-ordinates, which are

$$x = x' \cos \alpha + y' \cos \beta,$$

$$y = x' \sin \alpha + y' \sin \beta.$$

Thirdly, by making $\beta = \frac{\pi}{2} + \alpha$ we obtain formulæ for passing from an oblique to a rectangular system of co-ordinates, which are

$$x = \frac{x' \sin(\omega - \alpha)}{\sin \omega} - \frac{y' \cos(\omega - \alpha)}{\sin \omega},$$

$$y = \frac{x' \sin \alpha}{\sin \omega} + \frac{y' \cos \alpha}{\sin \omega}.$$

43. In the preceding transformations we have supposed the origin to remain unaltered; if, however, the origin is to be changed as well as the direction of the axes, we must employ the formulæ

$$x = x'' + h, \qquad y = y'' + k,$$

where h, k are the co-ordinates of the new origin parallel to the primitive axes, and x'', y'' denote the values of x and y, found in each of the preceding cases.

44. To transform rectangular into polar co-ordinates, and conversely.

Let $AN = x$, $PN = y$ (fig. 4) be the rectangular co-ordinates of any point P in a curve referred to the rectangular axes AX, AY; and let $AP = r$, $PAX = \theta$ be the polar co-ordinates of P.

$$\text{Then } AN = AP \cos PAN, \quad \text{or} \quad x = r\cos\theta,$$

$$PN = AP \sin PAN, \qquad y = r \sin \theta,$$

and any equation between x and y may be transformed into the polar equation by substituting these values for x and y.

Also, since $r = \sqrt{x^2 + y^2}$, and $\tan\theta = \frac{y}{x}$ or $\theta = \tan^{-1}\frac{y}{x}$, any equation between r and θ may be transformed to rectangular co-ordinates, by substituting these values of r and θ.

If the pole do not coincide with the origin of rectangular co-ordinates, then if h and k be its co-ordinates, the quantities to be substituted for x and y, in order to get the polar equation will be

$$x = h + r\cos\theta, \quad y = k + r\sin\theta.$$

SECTION IV.

ON THE CIRCLE.

Equation to a circle under various forms.

45. To find the equation to a circle when referred to two diameters at right angles to one another as axes.

Let P be any point in the circumference (fig. 18), $CN=x$, $NP=y$, its co-ordinates, and CP the radius equal c, then from the right-angled triangle CNP,

$$CN^2 + NP^2 = CP^2,$$

$$\text{or } x^2 + y^2 = c^2,$$

which is true for every point in the circumference, and is, therefore, the required equation; it expresses that the distance of every point from the origin is equal to c.

46. To find the equation to a circle when referred to any rectangular axes.

Let C be the center, and P any point in the circumference (fig. 19), draw CB, PN perpendicular to AX, and CM parallel to AX, and let the co-ordinates of C be $AB=a$, $BC=b$; the co-ordinates of P, $AN=x$, $NP=y$; and the radius $CP=c$.

Then from the right-angled triangle CPM,

$$CM^2 + MP^2 = CP^2.$$

But $CM = BN = x - a$, $MP = PN - CB = y - b$;

therefore we get for the general equation to the circle referred to rectangular axes,

$$(x-a)^2 + (y-b)^2 = c^2.$$

47. From this we may deduce several particular forms of the equation to a circle which are worthy of notice.

First, putting $a = 0$, $b = 0$, we have the origin in the center and fall upon the equation already found,

$$x^2 + y^2 = c^2.$$

Secondly, putting $a = c$, $b = 0$, we have the origin at the extremity of a diameter, and that diameter the axis of x, and we get

$$(x - c)^2 + y^2 = c^2,$$

or, reducing, $$y^2 = 2cx - x^2.$$

Thirdly, putting $b = 0$ simply, the center will be in the axis of x, but the origin will not be in the circumference; similarly, putting $a = 0$, the center will be in the axis of y, and in these two positions the equations will be

$$(x - a)^2 + y^2 = c^2,$$
$$x^2 + (y - b)^2 = c^2.$$

48. The above general equation to the circle when developed assumes the form

$$x^2 + y^2 + Ax + By + C = 0,$$

which does not contain the rectangle xy, and where the coefficients of the squares are equal. When these conditions are satisfied (the axes always being rectangular) the equation cannot represent any other curve except a circle. In fact by completing the squares we get

$$(x + \tfrac{1}{2}A)^2 + (y + \tfrac{1}{2}B)^2 = \tfrac{1}{4}A^2 + \tfrac{1}{4}B^2 - C,$$

which evidently represents a circle the ordinates of whose center are $-\tfrac{1}{2}A$, $-\tfrac{1}{2}B$; and of which the radius

$$= \sqrt{\tfrac{1}{4}A^2 + \tfrac{1}{4}B^2 - C}.$$

The equation however, will not in reality represent a circle, unless the quantity $\tfrac{1}{4}A^2 + \tfrac{1}{4}B^2 - C$ is positive; if this quantity is zero the circle is reduced to a point, namely the center; if it is negative the equation is impossible. Peculiarities of this sort are offered by the equations

$$x^2 + y^2 - 8y - 12x + 52 = 0,$$

$$x^2 + y^2 - 4y + 2x + 9 = 0,$$

which may be respectively reduced to the forms

$$(x-6)^2 + (y-4)^2 = 0, \quad (x+1)^2 + (y-2)^2 = -4.$$

But the equation $x^2 + y^2 + 4y - 4x - 8 = 0$, by completing the squares, becomes $(x-2)^2 + (y+2)^2 = 16$, which represents a circle whose radius is 4, and the co-ordinates of the center 2 and -2.

49. To find the equation to the circle referred to oblique axes.

Let, as before, $AB = a$, $BC = b$, $AN = x$, $NP = y$ (fig. 20) be the co-ordinates of the center, and of any point in the circumference of a circle referred to the oblique axes AX, AY, which form with one another an angle $= \omega$. Draw CM parallel to AX; then from the triangle CPM in which $\angle CMP = \pi - \omega$, and whose sides are respectively equal to $x - a$, $y - b$, and c the radius, we get

$$(x-a)^2 + (y-b)^2 + 2(x-a)\,.\,(y-b)\cos\omega = c^2,$$

the required equation.

50. The above equation when developed becomes

$$x^2 + y^2 + 2xy\cos\omega - 2(a + b\cos\omega)x - 2(b + a\cos\omega)y$$
$$+ a^2 + b^2 + 2ab\cos\omega - c^2 = 0,$$

which is of the form

$$x^2 + y^2 + 2xy\cos\omega + Ax + By + C = 0.$$

Whenever, therefore, an equation of the second order between oblique co-ordinates of known inclination, is such that the two squares x^2 and y^2 have unity for coefficients, and the rectangle xy has for coefficient twice the cosine of the angle between the axes, the equation represents a circle, the co-ordinates of whose center, a and b, and the radius c, are

determined by the equations $a + b\cos\omega = -\frac{1}{2}A$, $b + a\cos\omega = -\frac{1}{2}B$, $a^2 + b^2 + 2ab\cos\omega - c^2 = C$.

Also dropping the perpendiculars CD, CE (fig. 20) upon the axes, we get

$$AD = a + b\cos\omega = -\tfrac{1}{2}A, \quad AE = b + a\cos\omega = -\tfrac{1}{2}B.$$

If therefore we take the distances AD, AE equal respectively to half the coefficients of x and y with contrary signs, and erect the perpendiculars CD, CE, we shall determine the position of the center C.

Thus in order that the equation

$$x^2 + xy + y^2 - 2ax - 2ay + a^2 = 0$$

may represent a circle, we must have $2\cos\omega = 1$ or $\omega = \frac{1}{3}\pi$; and if x', y' be the co-ordinates of its center, and c its radius,

$$x' + y'\cos\omega = a, \quad y' + x'\cos\omega = a, \quad x'^2 + y'^2 + 2x'y'\cos\omega - c^2 = a^2;$$

$$\therefore \quad x' = y' = \frac{2a}{3} \text{ and } c = \frac{a}{\sqrt{3}}.$$

51. To find the polar equation to a circle.

Taking the pole for the origin, let a, b be co-ordinates of the center of the circle (fig. 21), c its radius, and $AP = r$, $XAP = \theta$, the polar co-ordinates of any point P, supposing the initial line to coincide with the axis of x.

Then substituting $r\cos\theta$ for x and $r\sin\theta$ for y in the equation

$$(x - a)^2 + (y - b)^2 = c^2 \text{ and expanding we have}$$

$$r^2\cos^2\theta - 2ar\cos\theta + a^2 + r^2\sin^2\theta - 2br\sin\theta + b^2 = c^2,$$

$$\text{or} \quad r^2 - 2(a\cos\theta + b\sin\theta)\,r + a^2 + b^2 - c^2 = 0,$$

the required equation.

52. If we suppose the pole to be in the circumference, and the initial line to be a diameter, we have $b=0$, $a=c$, and the equation becomes

$$r^2 - 2cr\cos\theta = 0;$$

$$\therefore \quad r = 2c\cos\theta,$$

at which we arrive immediately by joining AP, PB (fig. 18), for the right-angled triangle BAP gives

$$AP = AB\cos BAP, \quad \text{or} \quad r = 2c\cos\theta.$$

53. Let PY be a tangent to the circle at P (fig. 21), and AY a perpendicular upon it from the pole; and suppose $AY=p$, $AC=d$, then

$$\frac{p}{r} = \sin APY = \cos APC = \frac{r^2+c^2-d^2}{2rc},$$

or $2cp = r^2 + c^2 - d^2$ a relation between r and p.

If the pole be in the circumference, or $c=d$, this becomes

$$r^2 = 2cp.$$

54. From the preceding equations to the circle, which assume no other property of a circle than that it is the locus of a point which is always at the same distance from a given fixed point, all the theorems relative to the circle established in geometry may readily be deduced. We shall however confine our attention to those which relate to the tangent.

Tangent and Normal to a Circle.

55. To find the equation to a straight line which touches a circle at a proposed point.

In geometry a line is said to touch a circle when it has only one point in common with the circumference; if therefore through the two points P, P' (fig. 22), we draw a

secant PP', and then make it turn about P, till P' coincides with P, the secant in its ultimate position will become a tangent at P, for it will have only one point in common with the circumference. This consideration furnishes an easy method of determining the tangent at a given point of the circumference.

Let the co-ordinates of P, P', be respectively x', y'; x'', y'', and let α' be the angle which the line joining them makes with the axis of x,

$$\text{then } \tan\alpha' = \frac{y' - y''}{x' - x''};$$

but since the points are in the circumference, their co-ordinates must satisfy the equation to the circle;

$$\therefore \quad y'^2 + x'^2 = c^2, \quad y''^2 + x''^2 = c^2,$$

and subtracting the latter from the former we get

$$y'^2 - y''^2 + x'^2 - x''^2 = 0,$$

$$\text{which gives } \frac{y' - y''}{x' - x''} = -\frac{x' + x''}{y' + y''},$$

$$\text{and consequently } \tan\alpha' = -\frac{x' + x''}{y' + y''}.$$

Now let $x'' = x'$ and $y'' = y'$ so that P' coincides with P, and the secant PT' assumes the position of the tangent PT; therefore denoting by α the angle which the tangent forms with the axis of x, we get

$$\tan\alpha = -\frac{x'}{y'}, \text{ and } y - y' = -\frac{x'}{y'}(x - x')$$

for the equation to the tangent; or multiplying by y', and remembering that $x'^2 + y'^2 = c^2$, the equation to the line touching the circle at the point x', y' is in its most simple form

$$yy' + xx' = c^2.$$

56. The equation to CP is $y = \frac{y'}{x'}x$, which compared with the above equation shews that CP and PT are at right

angles to one another; that is, the tangent to a circle at any point, and the radius drawn to the point of contact, are perpendicular to one another.

57. To find the equation to the normal at any point of a circle.

A line drawn through the point of contact perpendicular to the tangent is called a normal. The co-ordinates of the point being x', y', the equation to the normal will be of the form

$$y - y' = m'(x - x'),$$

the condition of being perpendicular to the tangent gives

$$m' = -\frac{1}{m} = \frac{y'}{x'} \quad \text{(Art. 24)};$$

$$\therefore \quad y - y' = \frac{y'}{x'}(x - x'),$$

or reducing, $$y = \frac{y'}{x'}x,$$

which is the equation to a line passing through the origin, in this case the center. Hence all normals to a circle pass through the center.

58. To find the equation to a line which touches the circle, and passes through a given point without the circle.

Let h, k, be the co-ordinates of the given point; and x', y', those of the point of contact which are unknown; when they are found we shall have for the equation to the tangent

$$yy' + xx' = c^2,$$

and as the tangent passes through the given point, its equation must be satisfied by the co-ordinates of that point,

$$\therefore \; ky' + hx' = c^2,$$

and since the point of contact is in the circumference,

$$y'^2 + x'^2 = c^2,$$

which are the two equations that serve to determine x' and y'.

59. When a problem, as in the present case, leads to two equations between the co-ordinates x' and y' of an unknown point, each of the equations taken separately gives a geometrical locus on which the point is placed; consequently if we construct the two loci, we shall have two lines, the intersections of which will determine the points which satisfy the problem.

The locus of the second equation is the proposed circle, the locus of the first is a straight line AB (fig. 23), which is constructed by taking $CA = \frac{c^2}{h}$, $CB = \frac{c^2}{k}$, and joining AB; then the points T, T' in which this line cuts the circle are the points required.

Either of the equations may be replaced by another which results from combining them in any manner; and in solving problems in this way we must always select the combinations whose loci are easiest to construct.

Thus if we subtract the above equations we get

$$y'^2 - y'k + x'^2 - x'h = 0, \text{ or } (y' - \tfrac{1}{2}k)^2 + (x' - \tfrac{1}{2}h)^2 = \tfrac{1}{4}k^2 + \tfrac{1}{4}h^2,$$

which represents a circle whose center is O the middle point of CP, and radius CO, P being the point through which the tangents are to be drawn; if then we join CP and bisect it in O (fig. 23), and with center O and radius OC describe a circle cutting the former in T, T', these are the points of contact, and are determined by a simpler construction than the former one.

The value $AC = \frac{c^2}{h}$, which determines the point A, in which TT' meets the axis of x, is independent of the

ordinate k of P; therefore A will remain in the same position for all positions of P in the indefinite line PM. If therefore from every point of a straight line we draw tangents to a circle and join the points of contact, all the secants will intersect in the same point.

60. If we consider two circles represented by the equations

$$x^2 + y^2 = c^2, \quad (x - a)^2 + y^2 = c'^2,$$

so that the axis of x joins the centers, we may easily prove the co-ordinates of their points of intersection to be

$$x = \frac{a^2 + c^2 - c'^2}{2a},$$

$$y = \pm \frac{1}{2a} \sqrt{(a + c + c') \, . \, (a + c - c') \, . \, (a + c' - c) \, . \, (c + c' - a)},$$

from which all the common theorems in geometry, relative to the intersections and contacts of circles with one another may be deduced.

SECTION V.

ON THE DIFFERENT ORDERS OF CURVES; AND ON THE DIVISION OF CONIC SECTIONS, OR CURVES OF THE SECOND ORDER, INTO THREE SPECIES.

---

61. LINES are divided into orders according to the degree of their equations, the degree being determined by the sum of the indices of x and y in that term of the equation (which is supposed to contain no fractional or irrational term) where it is greatest.

The straight line is the line of the first order, being the locus of the equation of the first degree between two variables; the circle is a line of the second order, or curve of the second order (these terms being used indifferently), because its equation is of the second degree.

62. Curves of the second order are those whose equations involve the squares, or the simple product of the variables x and y, but no powers or products of them which are of higher dimensions. Hence the equation to curves of the second order under its most general form, or, which is the same thing, the general equation of the second order between two variables, is

$$ay^2 + bxy + cx^2 + dy + ex + f = 0,$$

which (as will be hereafter shewn) by giving a proper position and direction to the origin and axes of the co-ordinates, can always be reduced to one of the forms

$$Ay^2 + Bx^2 = C,$$

$$y^2 = Ax,$$

representing two distinct families of curves; the former those which have a center, the latter those which have not a center.

63. The center of a curve is a point such that all lines drawn through it and meeting the curve both ways, are bisected in it.

An axis of a curve is a line with respect to which the curve is symmetrically situated.

64. Of the curves represented by the equation

$$Ay^2 + Bx^2 = C,$$

the origin is the center, and the axes of the co-ordinates are axes.

For suppose P (fig. 24) to be a point in the curve having co-ordinates x', y', then the equation is satisfied by these values; and since it contains only even powers of x and y, it is also satisfied by the same values taken negatively; but if we produce PC to P' and make $CP' = CP$, the co-ordinates of P' are $-x'$ and $-y'$; therefore P' is a point in the curve, and PP' is a chord, and it is bisected in C; that is, every chord is bisected in C, and therefore C is the center of the curve. Also the curve is situated symmetrically with respect to the co-ordinate axes; for if in the equation we put $x = CN = x'$, we get for y two equal values with contrary signs, PN, $P_{\prime}N$; so that for every point situated above the axis of x, there will be a corresponding point situated at an equal distance below it. Similarly, for a given value $CM = y'$ of the ordinate, the equation furnishes two equal values with contrary signs, MP, MP'', of the abscissa. Hence both the co-ordinate axes bisect their ordinates at right angles, and the curve is situated symmetrically with respect to them, or they are axes of the curve.

65. In the equation to curves of the second order that have a center,

$$Ay^2 + Bx^2 = + C,$$

having taken care to make the second member positive, since the coefficients of the variables cannot be both nega-

tive together, we can have only two varieties of form; one with both coefficients positive, the other with one coefficient negative; so that the equation may assume the two forms

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

The curves represented by these are called respectively, the Ellipse, and the Hyperbola.

66. Of the curve represented by the equation

$$y^2 = Ax,$$

the origin is not the center, since the equation does not remain unaltered when x and y are changed into $-x$ and $-y$; and we shall see hereafter that it cannot have a center; also, since only an even power of y enters into the equation, the axis of x is an axis of the curve, but the axis of y is not an axis of the curve.

The equation may always be reduced to the form $y^2 = 4ax$, where a is a positive quantity; because if A be negative, we have only to change x into $-x$, the effect of which will be merely to reverse the position of the curve. Hence the second division of lines of the second order offers only one variety, which is called the Parabola.

These three species of curves, to one or other of which all lines of the second order belong, are called Conic Sections.

67. Instead of entering upon the discussion of the general equation of the second order (which may more conveniently be reserved for a more advanced part of the work), we shall now separately investigate the equation to each of the Conic Sections from a simple definition which embraces all of them, and thence determine their figures and properties.

68. Def. The locus of a point whose distances from a given fixed point and a straight line given in position, are always to one another in a constant ratio, is called a Conic Section.

Thus let S (fig. 25) be the given fixed point, and KX the line given in position, P a point such that joining SP and drawing PM perpendicular to KX, the distances SP, PM, are always to one another in an invariable ratio, then the locus of P is a Conic Section.

The point S is called the focus, and the line KX the directrix. Since SP may be always equal to PM, or always less than PM in a constant ratio, or always greater than PM in a constant ratio, there will be a distinct species of Conic Section corresponding to each of these cases; in the first case the locus of P is called the Parabola, in the second the Ellipse, and in the third the Hyperbola.

SECTION VI.

ON THE PARABOLA.

Various forms of the equation to the Parabola.

69. To find the equation to the parabola.

The parabola is the locus of a point, whose distance from a given point is always equal to its distance from a given fixed line.

Let KX (fig. 26) be the given fixed line, and S the given point, from which draw SX perpendicular to KX, and bisect it in A; then A is a point in the curve; and since the distance SX is known, let $AS = a$.

Draw Ay parallel to KX, and take A for the origin, and Ax, Ay, for the rectangular axes of the co-ordinates; and let P be a point in the parabola, and $AN = x$, $PN = y$, its co-ordinates; then drawing PM perpendicular to KX, and joining SP,

$$SP^2 = PM^2,$$

$$\text{or } PN^2 + SN^2 = XN^2,$$

$$\text{or } y^2 + (x-a)^2 = (x+a)^2;$$

$$\therefore\ y^2 = 4ax,$$

the equation required.

70. To trace the parabola by means of its equation (fig. 26).

Solving the equation, we get $y = \pm 2\sqrt{ax}$,

which shews, since for each positive value of x there are two equal values of y with contrary signs, that the axis of x is

an axis of the curve; and that the origin is a point in the curve, since $x=0$, gives $y=0$; and that no part of the curve is situated to the left of A, for a negative value of x makes y imaginary; but that as x increases from zero to infinity, y also increases from zero to $\pm\infty$. Moreover the curve is always concave toward its axis, otherwise it might be intersected in more than two points by a straight line, which is impossible. The parabola has only one vertex, namely, the point where it is met by the axis; and only one focus and directrix; and consists of two perfectly similar infinite branches Ax, Ax' upon the same side of the axis of y, and situated symmetrically with respect to the axis of x, to which they turn their concavities.

71. The double ordinate through the focus is called the latus rectum of the parabola; to find its length, making $x = AS = a$, we get

$$y^2 = 4a^2; \quad \therefore y = \pm 2a = SB \text{ or } SC;$$

and consequently $BC = 4a$.

Hence, if P be any point in the curve, we have

$$PN^2 = BC \times AN,$$

or, the square of the ordinate is equal to the rectangle of the latus rectum and corresponding abscissa, and consequently varies as the abscissa.

72. Let the origin be a point C (fig. 27) in the curve, and let the axis of the abscissæ be a line perpendicular to the axis of the parabola passing through C; and let $CM = x$, $MP = y$, be the co-ordinates of any point P, and $CB = h$, $AB = k$, those of the vertex; then

$$PN^2 = 4a \, . \, AN,$$

$$\text{or } (h-x)^2 = 4a(k-y),$$

$$\text{or } y = \frac{h}{2a}x - \frac{x^2}{4a}, \text{ since } h^2 = 4ak,$$

another form of the equation which is sometimes useful.

73. To express the distance of any point in the parabola from the focus, in terms of its abscissa.

$$SP^2 = SN^2 + NP^2$$
$$= (x-a)^2 + 4ax = (x+a)^2;$$
$$\therefore\ SP = x + a.$$

74. In expressing, as above, the distance of any point in the parabola from an assumed point, it is only when the latter coincides with the focus that the expression becomes rational.

For let x', y', be the co-ordinates of the assumed point; x, y, those of the point in the parabola, and d their distance; then

$$d^2 = (x-x')^2 + (y-y')^2 = x^2 - 2xx' + x'^2 + y^2 - 2yy' + y'^2,$$

but $y = 2\sqrt{ax}$, therefore d^2, and *a fortiori* d, cannot be rational in terms of x, unless the term $2yy'$ vanish, which gives $y' = 0$; then, replacing y^2 by its value,

$$d^2 = x^2 + 4ax - 2xx' + x'^2$$

which must be a perfect square;

$$\therefore\ 4x'^2 = 4(2a - x')^2,$$
$$x' = 2a - x',$$
$$x' = a,$$

which with the value $y' = 0$, indicates the focus; this then is the only point whose distance from any point in the curve can be expressed rationally in terms of the abscissa of that point. This is sometimes given as the definition of the focus.

75. To find the polar equation to the parabola, the the focus being the pole.

Let $SP = r$, angle $PSx = \theta$, (fig. 26);

$$\therefore\ SN = r\cos\theta \text{ and } x = a + r\cos\theta,$$

but $SP = a + x$;

$$\therefore\ r = 2a + r\cos\theta,$$

$$\text{or } r = \frac{2a}{1 - \cos\theta}.$$

Sometimes the angle of revolution is measured from that part of the axis which passes through the vertex, in which case, if $ASP = \theta'$, putting $\pi - \theta'$ for θ, we get

$$r = \frac{2a}{1 + \cos\theta'}.$$

Tangent and Normal to the Parabola.

76. To find the equation to the tangent of a parabola at a given point.

Let x', y', be the co-ordinates of the given point P (fig. 28), and x'', y'', those of another point P' in the parabola near the former; if we draw a secant through these two points, and denote by a' the angle which it forms with the axis of x, we have

$$\tan a' = \frac{y'' - y'}{x'' - x'}.$$

But the points being in the parabola, we have

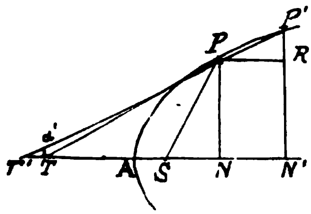
$$y''^2 = 4ax'', \quad y'^2 = 4ax';$$

$$\therefore\ y''^2 - y'^2 = 4a(x'' - x'),$$

$$\frac{y'' - y'}{x'' - x'} = \frac{4a}{y'' + y'},$$

$$\tan a' = \frac{4a}{y'' + y'}.$$

Now let P' move up to and coincide with P, then $x'' = x'$, $y'' = y'$, and the secant becomes the tangent at (x', y'); there-



$AN = x' : PN = y'.$

$AN' = x'' : P'N' = y''$

$$\tan \alpha' = \frac{PN}{T'N} = \frac{P'R}{PR} = \frac{y''-y'}{x''-x'}$$

fore, denoting by α the angle PTN which the tangent makes with the axis of x, we get

$$\tan \alpha = \frac{2a}{y'}. \qquad \frac{y-y'}{x-x'} = \frac{2a}{y'}$$

And the equation to the tangent is

$$y - y' = \frac{2a}{y'}(x - x')$$

or, multiplying by y' and observing that $y'^2 = 4ax'$,

$$yy' = 2a(x + x');$$

in which x', y', are the co-ordinates of the point of contact, and x, y, co-ordinates of any point in the tangent line.

77. If in the formula $\tan \alpha = \frac{2a}{y'}$ we make $y' = 0$, we find $\tan \alpha = \infty$, therefore the tangent at the vertex is perpendicular to the axis; also if we suppose y' to increase up to infinity, α decreases to zero; hence the tangent to the parabola continually tends to become parallel to the axis.

Hence, the equation of Art. 72. may be put under the form

$$y = x \tan \alpha - \frac{x^2}{4a}, \text{ if } \angle TCB = \alpha.$$

78. In the parabola, the subtangent, which is the distance between the foot of the ordinate to any point and the intersection of the tangent at that point with the axis, is double of the abscissa.

For $TN \times \tan PTN = PN$ (fig. 29),

$$\text{or } TN \times \frac{2a}{y'} = y'; \quad \therefore\ TN = \frac{y'^2}{2a} = 2x'.$$

This result may also be obtained by making $y = 0$ in the equation to the tangent; this gives $x = -x'$, and proves that the point T where the tangent meets the axis of x, is situated

to the left of A, and at a distance $AT = AN$. Hence, adding AN to AT, we have the subtangent $TN = 2AN$.

This furnishes a simple construction for drawing a tangent to a parabola at a given point.

If P be the given point, and AN, NP its co-ordinates, we have only to take in NA produced, $AT = AN$, and join TP, then TP is the tangent required.

79. In any curve, a line drawn through the point of contact perpendicular to the tangent is called a normal; and the distance between the foot of the ordinate, and the intersection of the normal with the axis of x, is called the subnormal.

In the parabola, the subnormal is equal to half the latus rectum; for if PG be perpendicular to PT since in the triangle TPG (fig. 29), PN is drawn from the right angle perpendicular to the opposite side,

$$NG \times TN = PN^2, \text{ or } NG \times 2x = 4ax;$$

$$\therefore\ NG = 2a.$$

80. This result may also be obtained by finding the equation to the normal at the point (x', y') of the parabola.

It will be of the form

$$y - y' = m'(x - x');$$

and since it is perpendicular to the tangent whose equation is

$$y - y' = \frac{2a}{y'}(x - x'),$$

$$m' = -\frac{1}{m} = -\frac{y'}{2a} \quad \text{(Art. 26)};$$

therefore the equation to the normal is

$$y - y' = -\frac{y'}{2a}(x - x');$$

now make $y = 0$, then $x - x' = 2a$,

$$\text{or } AG - AN = NG = 2a.$$

81. Since $ST = AS + AT = a + x$,

and $SG = SN + NG = x - a + 2a = a + x$,

we have $SP = ST = SG$ (Art. 73).

82. Draw Px' through P parallel to the axis, then

$$\angle tPx' = PTS = SPT, \quad \text{since } SP = ST;$$

$$\text{also } \angle GPx' = SPG.$$

Hence the tangent and normal at any point make equal angles with the focal distance of that point, and with a line drawn through it parallel to the axis.

83. These properties furnish a simple method of drawing a tangent to a parabola through a given point.

First, let the point be in the parabola, as P (fig. 29); join SP, and with center S and radius SP, describe a circle cutting the axis in T and G, then if PT and PG be joined, they are the tangent and normal at P.

84. Next, let the point be without the parabola, as T (fig. 30), and with center T and radius TS describe a circle cutting the directrix in two points M and M', through which draw two parallels to the axis, meeting the parabola in P, P', these are the points of contact; for the triangles MPT, TPS are equal in all respects, therefore $\angle TPM = TPS$, and PT is a tangent at P; similarly, TP' is a tangent P'.

85. This problem may be also solved by means of the equation to the tangent, as in the case of the circle.

Let h, k, be the co-ordinates of the given external point, and x', y', those of the unknown point of contact; then since they must satisfy both the equation to the tangent and that to the curve, we have to determine them;

$$ky' = 2a(h + x'),$$

$$y'^2 = 4ax',$$

and if we construct the straight line represented by the former, considering x' and y' as the variables, the points in which it intersects the parabola are the points of contact.

It appears that if a pair of tangents be drawn from an external point (h, k), the equation to the chord joining the points of contact is

$$ky = 2a(x + h).$$

86. The locus of the foot of the perpendicular dropped from the focus upon the tangent to a parabola is the line touching the parabola at its vertex.

Let PT (fig. 29) the tangent at P, meet Ay in Y, and join SY; then because TN is bisected in A, PT is bisected in Y; and since $SP = ST$, the triangles SPY, STY are equal in all respects; therefore SY is perpendicular to PT. Hence the tangent at any point and the perpendicular upon it from the focus intersect in the line which touches the parabola at the vertex.

87. Also from the right-angled triangle SYT we have

$$SY^2 = ST \times SA = SP \times SA,$$

or $p^2 = ar$, denoting SP, SY, by p and r respectively.

88. Let the tangent at P (fig. 31) meet the directrix in Q, draw PM perpendicular to the directrix, and join SQ; then $SP = PM$, PQ is common to the two triangles SPQ, QPM, and angle $SPQ = QPM$ by what has been proved;

$$\therefore \angle QSP = PMQ = \text{a right angle}.$$

Hence if a perpendicular to any focal distance SP intersect the directrix in Q, and QP be joined, QP is a tangent at P.

Therefore producing PS to meet the parabola in P' and joining QP', this is a tangent at P'; and since angle $PQS = PQM$ and angle $P'QS = P'QM'$, we have angle $PQP' = \frac{1}{2}\pi$. Hence the tangents at the extremities of any focal chord intersect at right angles in the directrix; and the line joining their point of intersection and the focus is perpendicular to the chord.

The Parabola referred to any Diameter.

89. To determine the intersection of a straight line with a parabola.

Let $y = mx + c$, be the equation to any straight line; if this line intersect a parabola whose equation is $y^2 = 4ax$, and if x', y', be the co-ordinates of a common point, we have

$$y' = mx' + c, \quad y'^2 = 4ax';$$

$$\therefore\ y' = \frac{m}{4a} y'^2 + c, \quad \text{or } y'^2 - \frac{4a}{m} y' + \frac{4ac}{m} = 0,$$

the roots of which are the ordinates of the points where the straight line meets the curve. Hence a straight line cannot cut a parabola in more than two points; if the roots be equal, the points of section coincide, and the line is then a tangent, and we have $\frac{16ac}{m} = \frac{16a^2}{m^2}$, or $m = \frac{a}{c}$; if the roots be impossible, the line falls entirely without the parabola.

90. To find the locus of the middle points of a system of parallel chords.

Let QQ' (fig. 32) be any chord whose equation is $y = mx + c$, V its middle point; draw VM perpendicular to Ax, then

$$VM = Q'N' + \tfrac{1}{2}(QN - Q'N') = \tfrac{1}{2}(Q'N' + QN),$$

but the values of QN, $Q'N'$, are the roots of the equation

$$y^2 - \frac{4a}{m} y + \frac{4ac}{m} = 0,$$

obtained by eliminating x between the equations to the parabola and chord;

$$\therefore\ QN + Q'N' = \frac{4a}{m};$$

hence denoting the ordinate of V by Y, we have for the equation to the locus of V, $Y = \frac{2a}{m}$, which represents a straight line PV parallel to the axis; and since the equation does not involve c, it bisects all chords for which m is the same, that is, all chords parallel to QQ'.

91. The line PV is called a diameter of the parabola. All diameters of a parabola are parallel to the axis, and intersect the corresponding chords at different angles varying from zero to a right angle.

Conversely, every straight line parallel to the axis may be considered as a diameter to the parabola; for by giving m a suitable value in the equation $Y = \frac{2a}{m}$, Y may become equal to any quantity we please.

92. Suppose a diameter Px' to be drawn at a distance y' from the axis, we have for this diameter

$$\frac{2a}{m} = y', \text{ or } m = \frac{2a}{y'}.$$

The quantity m is the tangent of the angle at which the diameter in question meets the chords which it bisects; it is also (Art. 76) the value of the tangent of the angle which the line touching the parabola at P, makes with the axis of x; therefore the chords bisected by any diameter, are parallel to the tangent at the extremity of that diameter.

The semi-chord QV is called an ordinate to the diameter PV.

In order that m may be infinite, we must have $y' = 0$; hence the axis of x is the only diameter which bisects its ordinates at right angles, or is the only axis of the parabola.

93. To find the equation to the parabola when referred to any diameter, and the tangent at the extremity of the diameter, as axes.

The formulæ for passing from rectangular to oblique axes are (Art. 42),

$$x = x' \cos a + y' \cos \beta + h, \quad y = x' \sin a + y' \sin \beta + k.$$

Suppose the new origin to be a point P (fig. 35) in the curve; then between h and k we have the relation $k^2 = 4ah$; also let the diameter Px' be the axis of x', then $a = 0$, and if we take the tangent Py' for the axis of y', we have $\tan \beta = \dfrac{2a}{k}$. Now substituting in the equation $y^2 = 4ax$, we get

$$(y' \sin \beta + k)^2 = 4a\,(x' + y' \cos \beta + h),$$

$$\text{or} \quad y'^2 \sin^2 \beta + 2y'k \sin \beta + k^2 = 4ax' + 4ay' \cos \beta + 4ah,$$

$$\text{or} \quad y'^2 \sin^2 \beta = 4ax', \quad \text{since} \quad k^2 = 4ah, \quad \text{and} \quad k \sin \beta = 2a \cos \beta.$$

$$\text{But} \quad \frac{a}{\sin^2 \beta} = a\,(1 + \cot^2 \beta) = a\left(1 + \frac{k^2}{4a^2}\right) = a + h = SP,$$

$$\therefore \quad y'^2 = 4SP\,.\,x' = 4a'x', \text{ if } SP = a', \text{ or } QV^2 = 4SP\,.\,PV.$$

94. The coefficient $4SP$, by which one diameter differs from another, is called the parameter of the diameter to which the parabola is referred; it is equal to four times the distance of the focus from the extremity of the diameter. It is also equal to the double ordinate passing through the the focus. For draw QQ' (fig. 33) through the focus S parallel to the tangent PT; then $PV = ST = SP$,

$$\therefore \quad QV^2 = 4SP \times PV = 4SP^2;$$

$$\therefore \quad QV = 2SP, \text{ and } QQ' = 4SP.$$

95. The equation to a parabola being of the same form when referred to a diameter and the tangent at its extremity, as when referred to its axis, the properties which are independent of the inclination of the co-ordinates, will be the same in the two systems. Hence the equation to the tangent at a point Q (x', y') will be

$$yy' = 2a'\,(x + x'),$$

where $\frac{2a'}{y'}$ denotes the ratio of the sines of the angles which the tangent makes with the axes of x and y; and when the tangent meets the axis of x we shall have $x = -x'$, or the subtangent $= 2x' =$ twice the abscissa of the point of contact, in all cases, i. e. $TV = 2\,PV$ (fig. 35).

96. Also if we wish to draw a tangent through an external point Q (h, k), we shall have, to determine the points of contact (x', y'), the equations

$$y'^2 = 4a'x', \quad y'k = 2a'(x' + h);$$

and if we construct the line represented by the latter, considering x' and y' as the variables, by taking $PT = -h$, $PR = \frac{2a'h}{k}$ (fig. 34), and joining TR, it will cut the parabola in the two points of contact.

97. Since the distance PT is independent of k, if through Q we draw a line parallel to Py', and from any other point in this line we draw a pair of tangents to the parabola, the secant passing through the new points of contact will cut the diameter Px' in T, as this point only changes when h changes. Hence if from the several points of a straight line, pairs of tangents be drawn to a parabola, the secants joining the corresponding points of contact will all intersect in the same point; and conversely, if through any point we draw different chords, and draw two tangents at the extremities of each, the locus of their intersection will be a straight line.

98. The tangents at the extremities of any chord will intersect in the diameter of which the chord is an ordinate.

For taking the diameter and the tangent at its extremity as axes, the equation to the tangent will be

$$\pm\, yy' = 2a'(x + x');$$

using the upper or lower sign according as we consider the point Q (x', y'), or the other extremity of the chord Q', whose

co-ordinates are x', $-y'$; and in both cases $y=0$ when $x=-x'$; therefore the tangents meet the diameter in the same point T (fig. 35).

99. Having given the parameter of any diameter of a parabola, and the inclination of the corresponding ordinates, to describe the parabola.

Let Px' be the given diameter (fig. 33), draw the line $y'PT$ at the given inclination to Px', this line will be a tangent to the parabola at the point P. Make the angle $TPS = y'Px'$, and PS a quarter of the given parameter; then S will be the focus. Take $PM = PS$ and draw ML perpendicular to Mx', this will be the directrix; and knowing the focus and directrix, the parabola can of course be described.

100. If a parabola be traced upon a plane, we may determine its axis by drawing two parallel chords QQ', qq', (fig. 35), and drawing a line VV' through their middle points, this will be a diameter. And if we draw any chord QR perpendicular to it, and through the middle point of QR draw AN parallel to VV', this will be the axis of the parabola.

101. If through any point within or without a parabola, two lines be drawn parallel to two given straight lines to meet the curve, the rectangles of the segments will be to one another in an invariable ratio.

Let O be the given point (fig. 36), Qq a line drawn through it in a known direction, and therefore an ordinate to a given diameter PV; draw the diameter $A'O$, and $A'v$ parallel to Qq; then

$$QO \times Oq = QV^2 - VO^2 = 4SP \times PV - 4SP \times Pv = 4SP \times A'O.$$

Similarly, if $Q'q'$ be an ordinate to the diameter whose extremity is P', passing through O,

$$Q'O \times Oq' = 4SP' \times A'O;$$

$$\therefore\ QO \times Oq : Q'O \times Oq' :: SP : SP',$$

a ratio independent of the position of O.

102. If the point O be without the parabola, and we suppose Q, q, to coincide in P, and Q', q' to coincide in P', so that OP, OP' become tangents, we have

$$OP^2 : OP'^2 :: SP : SP'.$$

103. Any parabolic segment ANP is two-thirds of the parallelogram whose sides are the abscissa AN and the ordinate NP.

Let NP, $N'P'$ (fig. 37) be ordinates to the diameter AN; at P, P' draw tangents meeting the diameter in T, T', and one another in R. Join PP' and draw RK parallel to AN, meeting PP' in I, and bisecting it in that point (Art. 98); and draw KH perpendicular to AN.

$$\text{Then area of triangle } TRT' = \tfrac{1}{2} TT' \times KH,$$

$$\text{area of trapezium } N'P'PN = NN' \times KH = TT' \times KH,$$

$$\text{since } AN = AT, \quad \text{and } AN' = AT'.$$

Hence the trapezium is double of the triangle. Similarly we may shew that trapezium $N''P''P'N'$ is double of the triangle $T'R'T''$, and so on. Hence the sum of the trapeziums is double of the sum of the triangles, and therefore the parabolic segment APN, which is the limit of the first sum, is double of the exterior segment APT, which is the limit of the second sum. Hence the segment ANP is two-thirds of the triangle TNP, or two-thirds of the parallelogram contained by AN, NP.

SECTION VII.

ON THE ELLIPSE.

Various forms of the equation to the Ellipse.

104. To find the equation to the Ellipse.

The ellipse is the locus of a point, whose distance from a given point is always less than its distance from a given fixed line, in a constant ratio.

Let S (fig. 38) be the given point, and KK' the given fixed line; and from S let fall the perpendicular SX upon KK'. Let P be a point in the ellipse; join SP and draw PM perpendicular to KX; and let the constant ratio $SP : PM$ be $e : 1$, e being less than 1. Divide SX in A so that $SA : AX :: e : 1$, then A is a point in the ellipse; and since the distance SX is known, we may assume $AS = p$, therefore $AX = \frac{p}{e}$. Through A draw Ay parallel to KX, and take A for the origin, and Ax, Ay, for the axes of the co-ordinates; and let $AN = x$, $PN = y$, be the co-ordinates of P.

$$\text{Then } SP^2 = e^2 . PM^2,$$

$$\text{or } SN^2 + NP^2 = e^2 . XN^2,$$

$$\text{or } (x - p)^2 + y^2 = e^2 \left(\frac{p}{e} + x\right)^2 = (p + ex)^2;$$

$$\therefore y^2 = 2p(1 + e)x - (1 - e^2)x^2$$

$$= (1 - e^2) . \left(\frac{2p}{1 - e}x - x^2\right),$$

or, if we replace the known quantity $\frac{p}{1-e}$ by a,

$$y^2 = (1 - e^2)(2ax - x^2),$$

the required equation.

105. To determine the points where the curve cuts the axis of x, make $y = 0$, then $x = 0$ or $x = 2a$; the value $x = 0$ gives the point A, already known, the other value $x = 2a = AA'$, determines the point A'.

Bisect AA' in C, then making in the equation to the ellipse $x = AC = a$, we get

$$y^2 = (1 - e^2)a^2, \quad \text{or } y = \pm a\sqrt{1 - e^2}.$$

If therefore through C we draw BB' perpendicular to AA' and take $CB = CB' = a\sqrt{1 - e^2}$; B, B', are points in the ellipse; and denoting BB' by $2b$, we have $a\sqrt{1 - e^2} = b$ and $1 - e^2 = \frac{b^2}{a^2}$, and the equation to the ellipse becomes

$$y = \pm \frac{b}{a}\sqrt{2ax - x^2}.$$

106. In order to transfer the origin to C the middle point of AA', we must change x into $a + x'$; since

$$AN = AC + CN = a + x';$$

$$\therefore\ y^2 = \frac{b^2}{a^2}\{2a(a + x') - (a + x')^2\} = \frac{b^2}{a^2}(a^2 - x'^2).$$

This form of the equation shews that the origin is the center of the ellipse, and that the axes of the co-ordinates are axes of the ellipse (Art. 64); but the term axis is more particularly appropriated to the portions of those lines, $AA' = 2a$, $BB' = 2b$, which fall within the curve; of these the greater (which passes through the foci, as we shall see) is called the major axis, and sometimes the transverse axis; and the other the minor, or conjugate axis. Their ex-

tremities A, A', B, B', are called the vertices of the ellipse, and their intersection, as has been said, the center.

107. To trace the ellipse by means of its equation.

The equation to the ellipse referred to its axes is

$$y = \pm \frac{b}{a}\sqrt{a^2 - x^2}.$$

Hence as x increases positively from zero to a, the two values of y are real and diminish from b to zero, and give the portion of the curve $BA'B'$ (fig. 38); but when x exceeds a, the values of y become imaginary, and therefore no part of the curve lies beyond A'; also the curve in every one of its points has its concavity turned towards the center, otherwise it might be cut by a straight line in more than two points, which is impossible. Hence, since the portion of the curve situated to the left of BB' is precisely similar and equal to the portion situated to the right, the shape of the curve will be that of the oval $ABA'B'$, surrounding the center on every side.

108. Since the ellipse is symmetrically situated with respect to the axes AA', BB', if we take $CH = CS$, $Cx = CX$, and draw kx perpendicular to AA', there is no reason why the curve may not be described by means of the focus H and directrix kx, just as well as by means of S and KX. Hence the ellipse has two foci S and H, equidistant from C.

$$\text{Also since } a = \frac{p}{1-e}, \quad \text{we have } AS = p = a(1-e);$$

$$\therefore\ SC = a - a(1-e) = ae, \quad \text{and } e = \frac{SC}{AC}.$$

109. The quantity e, which expresses the ratio of the distance between either focus and the center, to the semiaxis major, is called the eccentricity.

Since $b = a\sqrt{1-e^2}$, the eccentricity, expressed by the semi-axes, is equal to $\dfrac{\sqrt{a^2-b^2}}{a}$. Hence $SC = \sqrt{a^2-b^2}$, and $\therefore BS = a$; and if with center B and radius equal to the semi-axis major we describe a circle, it will intersect the major axis in the foci.

Hence each focus divides the major axis into the segments $a - \sqrt{a^2-b^2}$, $a + \sqrt{a^2-b^2}$, the rectangle of which equals the square of the semi-axis minor.

110. Since $AS = e \, . \, AX$, we have $AX = \dfrac{a(1-e)}{e}$;

$$\therefore CX = \frac{a}{e} = \frac{a^2}{ae} = \frac{CA^2}{CS},$$

which determines the directrix relative to the center.

111. The double ordinate passing through the focus is called the latus rectum.

To find its value, make $x = CH = ae$ in the equation

$$y^2 = \frac{b^2}{a^2}(a^2 - x^2),$$

$$\text{then } y^2 = \frac{b^2}{a^2}(a^2 - a^2e^2) = b^2(1-e^2) = \frac{b^4}{a^2};$$

$$\therefore y = \pm\frac{b^2}{a};$$

$$\therefore LL' = \frac{2b^2}{a}, \text{ or } = 2a(1-e^2).$$

112. When the distance CS between the focus and center becomes infinite, the distance AS between the focus and vertex remaining finite, the ellipse is changed into a parabola. For since $\dfrac{b^2}{a^2} = 1 - e^2$, the equation reckoned from the vertex may be written

$$y^2 = 2a(1 - e^2)x - (1 - e^2)x^2,$$

or $y^2 = 2p(1 + e)x - (1 - e^2)x^2$, if $AS = p$.

But $e = \dfrac{SC}{AC} = \dfrac{SC}{p + SC} = \dfrac{1}{1 + \dfrac{p}{SC}}$; let $SC = \infty$; $\therefore e = 1$;

$\therefore y^2 = 4px$, the equation to the parabola.

113. When $a = b$ the equations to the ellipse become

$$y^2 = a^2 - x^2, \quad y^2 = 2ax - x^2,$$

which represent a circle; hence when its axes are equal, the ellipse becomes a circle.

Upon the major axis as diameter describe a circle (fig. 39), and produce the ordinate NP of the ellipse to meet it in Q; then making $CN = x$, $NP = y$, $NQ = y'$, we have

$$y = \frac{b}{a}\sqrt{a^2 - x^2}, \quad y' = \sqrt{a^2 - x^2};$$

$$\therefore y = \frac{b}{a}y';$$

which shews that, corresponding to the same abscissa, the ordinates of the ellipse are to the ordinates of the circle in the constant ratio of the smaller to the larger axis; consequently the ellipse may be described by diminishing all the ordinates of the circle in that ratio.

114. Since $y^2 = \dfrac{b^2}{a^2}(a^2 - x^2) = \dfrac{b^2}{a^2}(a + x) \cdot (a - x)$,

gives $PN^2 = \dfrac{b^2}{a^2} \times A'N \times AN$ (fig. 39),

we see that the square of the ordinate is to the rectangle of the corresponding segments of the major axis, as the square of the semi-axis minor to the square of the semi-axis major; and that, consequently, the square of the ordinate varies as the rectangle of the corresponding segments of the major axis.

115. To express the distances of any point in the ellipse from the foci, in terms of its abscissa

$$SP^2 = SN^2 + PN^2 \text{ (fig. 39)}$$

$$= (ae + x)^2 + (1 - e^2) . (a^2 - x^2)$$

$$= (a + ex)^2;$$

$$\therefore SP = a + ex,$$

$$HP^2 = (ae - x)^2 + (1 - e^2) . (a^2 - x^2)$$

$$= (a - ex)^2;$$

$$\therefore HP = \pm (a - ex),$$

and since e is less than 1, and x is always less than a, in order that HP may be positive we must take the upper sign which gives

$$HP = a - ex.$$

116. In expressing, as above, the distance of any point in the ellipse from an assumed point, it is only when the latter coincides with one of the foci, that the expression becomes rational.

For let x', y', be the co-ordinates of the assumed point, x, y, those of any point in the ellipse, and d their distance,

$$\text{then } d^2 = (x - x')^2 + (y - y')^2$$

$$= x^2 - 2xx' + x'^2 + y^2 - 2yy' + y'^2.$$

But $y = \dfrac{b}{a}\sqrt{a^2 - x^2}$, therefore d^2, and *a fortiori* d, cannot be expressed rationally in terms of x unless the term $2yy'$ vanish, which gives $y' = 0$; then replacing y^2 by its value we get

$$d^2 = \left(1 - \frac{b^2}{a^2}\right) x^2 - 2xx' + x'^2 + b^2$$

which must be a perfect square;

$$\therefore\ 4\left(1-\frac{b^2}{a^2}\right)(x'^2+b^2)=4x'^2,$$

$$\text{or } x'=\pm\sqrt{a^2-b^2}.$$

These values require that a should be greater than b, i. e. that the abscissæ should be measured along the axis major; and with $y'=0$, they determine, as we perceive, the two foci S and H. These then are the only points whose distances from any point of the curve can be expressed rationally in terms of the abscissa of the point. This is sometimes given as the definition of the foci.

117. Hence, by addition we get

$$SP+HP=2a,$$

which expresses the remarkable property, that the sum of the focal distances of any point in the ellipse is constant, and equal to the major axis.

This affords a very simple method of determining any number of points of an ellipse of which we know the foci and axis major; in AA' take any point F (fig. 40), and with center S and radius equal to $A'F$ describe a circle; next with center H and radius equal to AF describe another circle, cutting the former in P, P', these are manifestly points in the ellipse.

When the ellipse is to be very large, we may describe it by fastening the ends of a cord equal to the axis major in the foci; then if we make a pole slide along the cord, so as to keep it stretched, the ellipse will be traced out by the extremity of the pole.

This property also furnishes a method of investigating the equation to the ellipse which is sometimes employed as follows.

118. To find the locus of a point, the sum of whose distances from two given points is constant.

Through the two fixed points S, H (fig. 40), draw the indefinite line Sx; bisect SH in C, and draw yC perpendicular to it, and take Cx, Cy, for the axes of the co-ordinates, as the locus of P will evidently be symmetrical with respect to these lines. Let $SC = CH = c$, $CN = x$, $NP = y$, the co-ordinates of any point P, and $SP + PH = 2a$;

$$\text{then } SP^2 = (x + c)^2 + y^2,$$

$$HP^2 = (x - c)^2 + y^2;$$

$$\therefore\ SP^2 - HP^2, \text{ or } 2a(SP - HP) = 4cx;$$

$$\therefore\ SP - HP = \frac{2cx}{a},$$

$$\text{but } SP + HP = 2a;$$

$$\therefore\ SP = a + \frac{cx}{a};$$

$$\therefore\ \left(a + \frac{cx}{a}\right)^2 = (x + c)^2 + y^2,$$

$$\text{or } y^2 = \frac{a^2 - c^2}{a^2}(a^2 - x^2).$$

Now $SP + HP$ is greater than SH, or a is greater than c, therefore $a^2 - c^2$ is a positive quantity; hence comparing the above with the equation to an ellipse

$$y^2 = \frac{b^2}{a^2}(a^2 - x^2),$$

we see that the equations are identical, and consequently so are the curves which they represent, if $b^2 = a^2 - c^2$; therefore the required locus is an ellipse whose major axis is $2a$, and minor axis $2\sqrt{a^2 - c^2}$.

119. To find the polar equation to the ellipse, one of the foci being the pole.

$$\text{Let } SP = r,\ ASP = \theta \text{ (fig. 40)};\ \therefore\ SN = r\cos\theta,$$

$$\text{and } CN = r\cos\theta - ae, \quad \text{but } SP = a + ex;$$

$$\therefore\ r = a + e\,(r\cos\theta - ae) = a\,(1 - e^2) + er\cos\theta\,;$$

$$\therefore\ r\,(1 - e\cos\theta) = a\,(1 - e^2),$$

$$\text{or } r = \frac{a\,(1 - e^2)}{1 - e\cos\theta}.$$

We have measured the angle θ from that part of the axis major which passes through the more remote vertex; sometimes it is measured from the nearer vertex A', in which case if $A'SP = \theta'$, putting $\pi - \theta'$ for θ we get

$$r = \frac{a\,(1 - e^2)}{1 + e\cos\theta'}.$$

Of course if the other focus H be taken for the pole, the formulæ will be exactly the same.

120. To find the polar equation to the ellipse, the center being the pole.

Let $CP = r$, $ACP = \theta$ (fig. 40), and let x and y be the rectangular co-ordinates of P; then $x = r\cos\theta$, $y = r\sin\theta$; therefore substituting these values in the equation

$$a^2y^2 + b^2x^2 = a^2b^2, \text{ we get}$$

$$r^2\,(a^2\sin^2\theta + b^2\cos^2\theta) = a^2b^2,$$

or, dividing by a^2 and observing that $\dfrac{b^2}{a^2} = 1 - e^2$,

$$r^2\,(1 - e^2\cos^2\theta) = b^2\,;$$

$$\therefore\ r = \frac{b}{\sqrt{1 - e^2\cos^2\theta}}.$$

In this formula it is indifferent from which vertex the angle θ is measured; it shews that of all lines drawn from the center to the curve, the semi-axis major is the greatest when $\theta = 0$, and the semi-axis minor the least when $\theta = \dfrac{\pi}{2}$.

Tangent and Normal to the Ellipse.

121. To find the equation to the tangent of an ellipse at a given point.

As in former cases, we regard the tangent as a secant which passes at first through two points of the curve, and then turns about one of them till the other moves up to and coincides with it.

Let x', y' be the co-ordinates of the given point in the curve, and x'', y'' those of another point in the curve near the former, and let α' be the angle which the line joining them makes with the axis of x.

$$\text{then } \tan\alpha' = \frac{y' - y''}{x' - x''}.$$

But the two points being in the ellipse, their co-ordinates must satisfy its equation;

$$\therefore\ a^2 y'^2 + b^2 x'^2 = a^2 b^2, \quad a^2 y''^2 + b^2 x''^2 = a^2 b^2.$$

Subtracting the latter from the former we get

$$a^2 (y'^2 - y''^2) + b^2 (x'^2 - x''^2) = 0,$$

$$\text{which gives } \tan\alpha' = \frac{y' - y''}{x' - x''} = -\frac{b^2}{a^2} \cdot \frac{x' + x''}{y' + y''}.$$

Now let the second point move up to and coincide with the first, then $x'' = x'$, $y'' = y'$ and the secant becomes a tangent at (x', y'); therefore, denoting by α the angle which the tangent makes with the axis of x, we get

$$\tan\alpha = -\frac{b^2 x'}{a^2 y'};$$

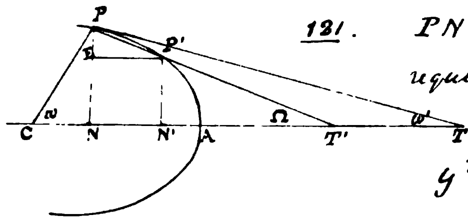
therefore the equation to the tangent will be

$$y - y' = -\frac{b^2 x'}{a^2 y'} \cdot (x - x'),$$

or under another form, recollecting that $a^2 y'^2 + b^2 x'^2 = a^2 b^2$,

$$a^2 y y' + b^2 x x' = a^2 b^2.$$

122. The formula $m = \tan\alpha = -\frac{b^2}{a^2} \cdot \frac{x'}{y'}$, since it does not change when x' and y' are replaced by $-x'$ and $-y'$,



121. $PN = y$: $CN = x$: $P'N' = y'$: $CN' = x'$

required tan. PT'.

$$\tan\Omega = \tan PP'E = \frac{y-y'}{x'-x}$$

$$y^2 = \frac{b^2}{a^2}(a^2 - x^2)$$
$$= \frac{b^2a^2 - b^2x^2}{a^2}$$
$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$
$$\therefore \frac{y^2}{b^2} + \frac{x^2}{a^2} = 1 \quad \text{--- } 1$$
$$\& \quad \frac{y'^2}{b^2} + \frac{x'^2}{a^2} = 1 \quad \text{--- } 2$$

subtracting 1 fr. 2.

$$\frac{(x'-x)(x'+x)}{a^2} + \frac{(y'-y)(y'+y)}{b^2} = 0$$
$$b^2(x'-x)(x'+x) + a^2(y'-y)(y'+y) = 0$$
$$-(y'-y)(y'+y) = \frac{b^2}{a^2}(x'-x)(x'+x)$$
$$\therefore \frac{y-y'}{x'-x} = \frac{b^2}{a^2} \cdot \frac{x'+x}{y'+y}$$
$$\tan\Omega = \frac{b^2}{a^2} \cdot \frac{x'+x}{y'+y}$$

In the limit Ω becomes ω'
and, $x' = x$, $y' = y$

$$\therefore \tan\omega' = \frac{b^2}{a^2}\frac{2x}{2y} = \frac{b^2}{a^2}\frac{x}{y} -$$

And the product of $\tan\omega . \tan\omega'$ is always a constant. for

$\tan\omega = \frac{y}{x}$: $\tan\omega' = \frac{b^2}{a^2} \cdot \frac{x}{y}$

$$\therefore \tan\omega . \tan\omega' = \frac{b^2}{a^2} = (1 - e^2)$$

shews that if PC be produced to meet the ellipse in P' (fig. 39) the tangents at P and P' are parallel, as we might have foreseen from the symmetrical position of the ellipse, relative to the axes; also it proves that at the points B, B', for which $x'=0$, $y'=\pm b$, $\tan\alpha=0$, or the tangents are parallel to AA'; and at A, A', for which $x'=\pm a$ and $y'=0$, the tangents are perpendicular to AA'; and that for intermediate points, going from A to B, the angle PTx which is always obtuse, continually increases till PT becomes parallel to AA' at B.

123. To find where the tangent meets the axis major, put $y=0$, then $x=\dfrac{a^2}{x'}$, or $CT=\dfrac{CA^2}{CN}$. As this result is independent of b, it will be the same for all ellipses constructed upon AA' as an axis; consequently if NP meet the circle whose diameter is AA' in Q, the tangent to the circle at Q will pass through T.

124. Subtracting CN or x' from the value just found for CT, we get the subtangent

$$NT=\frac{a^2}{x'}-x'=\frac{a^2-x'^2}{x'}.$$

125. To find the equation to the normal at any point of an ellipse.

It will be of the form

$$y-y'=m'(x-x'),$$

and since it is perpendicular to the tangent,

$$m'=-\frac{1}{m}=\frac{a^2y'}{b^2x'}.$$

therefore the equation to the normal is

$$y-y'=\frac{a^2y'}{b^2x'}\cdot(x-x').$$

126. To determine the point G, where the normal meets the axis major.

Make $y = 0$.

$$\therefore \quad x = x'\left(1 - \frac{b^2}{a^2}\right),$$

$$\text{or} \quad CG = e^2 . CN \quad \text{(fig. 41)},$$

$$\text{and the subnormal } GN = \frac{b^2}{a^2} . CN.$$

127. The normal at any point bisects the angle between the focal distances of that point.

$$\text{For} \quad SG = ae + e^2x' = e(a + ex') = e . SP,$$

$$\text{and} \quad HG = ae - e^2x' = e(a - ex') = e . HP.$$

$$\therefore \quad \frac{SG}{HG} = \frac{SP}{HP},$$

and consequently the normal PG bisects the angle SPH (Euc. vi. 3). Also drawing the tangent YZ at P, since $GPY = GPZ$, each of them being a right angle, and $GPS = GPH$; therefore the angle $SPY = HPZ$, or the focal distances make equal angles with the tangent on the same side of it.

128. These properties furnish a simple method of drawing a tangent to an ellipse through a given point.

First let the point be in the ellipse as P (fig. 42); join SP, HP, and produce SP to K, making $SK = 2AC$; join HK and draw PZ perpendicular to it, then PZ is a tangent at P.

For $PK = 2AC - SP = PH$, PZ is common, and the angles at Z are right angles, therefore angle

$$HPZ = KPZ = SPY,$$

and consequently PZ is a tangent at P.

129. Next let the point be without the ellipse.

With center S (fig. 43), and radius $= 2AC$ describe a circle KK', and with center T, the given external point, and radius TH, describe a circle cutting the former in K, K'; join SK meeting the ellipse in P, and join TP; then TP is a tangent at P; for $PK = PH$, $TK = TH$, and TP is common, therefore TP bisects the exterior angle HPK and is a tangent at P; similarly if SK' be joined meeting the the ellipse in P', a second point of contact will be found.

There is no difficulty in shewing that as long as T is exterior to the ellipse, the circles must intersect in two points.

130. The locus of the feet of the perpendiculars dropped from the foci upon the tangent to an ellipse, is the circumference of the circle whose diameter is the axis major.

For joining CZ (fig. 42), since SH is bisected in C and HK in Z, CZ is parallel to SK and equal to $\frac{1}{2} SK = AC$. Also drawing SY, CQ parallel to HZ, CQ bisects YZ perpendicularly, and therefore $CY = CZ = AC$.

131. Since C is the center of the circle which is the locus of Y and Z, and SYZ is a right angle, if YS and ZC be produced to meet in H', this will be a point in the circumference of the circle; and $SH' = HZ$.

hence $SY \times HZ = SY . SH' = AS \times A'S = BC^2$ (Art. 109).

132. Since $SY \times HZ = BC^2$,

and $\dfrac{HZ}{HP} = \dfrac{SY}{SP}$, or $\dfrac{SY}{HZ} = \dfrac{SP}{HP}$ (Art. 127);

therefore multiplying together these equations

$$SY^2 = BC^2 \times \frac{SP}{HP};$$

or if SY be denoted by p and SP by r, between the radius vector and the perpendicular on the tangent from the focus we have the relation

$$p^2 = b^2 . \frac{r}{2a - r}.$$

133. Draw HI parallel to YZ, and suppose the angle $SPY = a$,

$$\text{then } \tan STP = \tan SHI = \frac{SI}{IH} = \frac{r \sin a - (2a - r) \sin a}{r \cos a + (2a - r) \cos a}$$

$$= \frac{2(r - a)}{2a} \tan a,$$

$$\text{or } \tan STP = \left(\frac{SP}{AC} - 1\right) \tan SPY;$$

a result which is sometimes of use.

134. Draw CE parallel to PY, then PC is a parallelogram,

$$\text{and } PE = CZ = AC.$$

135. If from G the foot of the normal at P (fig. 41), GL be drawn perpendicular to either focal distance, then $PL = \frac{1}{2}$ the latus rectum.

For by similar triangles, $\dfrac{SL}{SG} = \dfrac{SN}{SP}$;

$$\therefore \quad SL = SN \cdot \frac{SG}{SP} = e(ae + x) \quad \text{(Art. 127)};$$

$$\therefore \ PL = a + ex - (ae^2 + ex) = a(1 - e^2) = \text{half the latus rectum.}$$

136. If a perpendicular be dropped from the center upon the tangent at any point of an ellipse, making an angle ϕ with the axis major, its length $= a\sqrt{1 - e^2 \sin^2 \phi}$.

Let CQ, HZ (fig. 44), be perpendiculars upon PT the tangent at any point P from the center and focus; join CZ and let $TCQ = \phi$,

then $CZ = a$, and $QZ = CH \sin \phi = ae \sin \phi$.

$$\therefore \ CQ^2 = a^2 - a^2 e^2 \sin^2 \phi,$$

$$\text{or } CQ = a\sqrt{1 - e^2 \sin^2 \phi}.$$

137. Let $P'T'$ (fig. 44) be another tangent to the ellipse at right angles to the former, and CQ' perpendicular to it, then $Q'CT' = \frac{1}{2}\pi - \phi$;

$$\text{and therefore } CQ'^2 = a^2 - a^2e^2\cos^2\phi.$$

$$\text{Hence } CW^2 = CQ^2 + CQ'^2 = a^2 + a^2(1 - e^2) = a^2 + b^2,$$

and therefore the locus of the intersection of two tangents to an ellipse at right angles to one another is a circle whose center is C and radius equal to $\sqrt{a^2 + b^2}$.

The Ellipse referred to its conjugate Diameters.

138. To find the locus of the middle points of a system of parallel chords.

Let the chords be parallel to a line CD (fig. 45) through the center, whose equation is $y = mx$; then the equation to any one of them QQ' will be

$$y = mx + c,$$

and to determine the points in which it meets the ellipse, we must combine its equation with the equation to the ellipse

$$a^2y^2 + b^2x^2 = a^2b^2,$$

this gives, eliminating x by the substitution $\frac{1}{m}(y - c)$,

$$y^2 - \frac{2b^2c}{a^2m^2 + b^2}y + \frac{b^2(c^2 - m^2a^2)}{a^2m^2 + b^2} = 0,$$

the roots of which are represented by QM, $Q'M'$; but if V be the middle point QQ', and $CN = X$, $VN = Y$ its coordinates,

$$Y = \frac{1}{2}(QM + Q'M') = \frac{b^2c}{m^2a^2 + b^2},$$

$$X = \frac{1}{m}(Y - c) = -\frac{ma^2c}{m^2a^2 + b^2};$$

therefore, dividing one result by the other in order to eliminate the quantity c, which particularizes the chord, we get

$$Y = -\frac{b^2}{ma^2} . X,$$

a relation between the co-ordinates of the middle point of any chord, and therefore the equation to its locus, which is consequently a straight line PP' passing through the origin.

The straight line which passes through the middle points of a system of parallel chords is called a diameter. Hence all diameters of an ellipse pass through its center; and, conversely, every line through the center is a diameter.

139. Hence denoting the equation to any chord by

$$y = mx + c,$$

and the equation to the diameter which bisects it by $y = m'x$, we have

$$m' = -\frac{b^2}{ma^2}, \quad \text{or } mm' = -\frac{b^2}{a^2},$$

a simple relation by means of which the equation of one may always be deduced from that of the other.

140. The chords which are parallel to PP' are bisected by CD.

For any one of these chords RR' may be represented by the equation

$$y = m'x + c',$$

then the diameter which bisects them, by what has been proved, will have for its equation

$$y = -\frac{b^2}{m'a^2} . x, \quad \text{or } y = mx, \quad \text{since } mm' = -\frac{a^2}{b^2},$$

which belongs to CD.

Hence two diameters, whose equations $y = mx$, $y = m'x$ are so related, that $mm' = -\dfrac{b^2}{a^2}$, have the property that

each bisects the chords parallel to the other; they are called conjugate diameters, or rather those portions of them PP', DD', which fall within the ellipse are usually called a pair of conjugate diameters.

141. If PT be the tangent at P, and x', y', the co-ordinates of P, then $m = \frac{y'}{x'}$, therefore the equation to PT is

$$y - y' = -\frac{b^2}{ma^2} \cdot (x - x')$$

which represents a line parallel to CD.

Hence the tangent applied at the extremity of any diameter is parallel to the corresponding conjugate diameter; and if tangents be drawn at the extremities of a pair of conjugate diameters, they will form a parallelogram circumscribing the ellipse, (Art. 122.)

142. Having given the co-ordinates of the extremity of any diameter, to find those of the extremity of the diameter conjugate to it.

Let x', y', be the co-ordinates of the point P (fig. 46),

$$\text{then } y = \frac{y'}{x'}\, x \text{ is the equation to } CP,$$

$$\text{and } \therefore\ y = -\frac{b^2 x'}{a^2 y'}\, x \text{ the equation to } CD.$$

To determine the co-ordinates of D, we must combine the equation to CD with the equation to the ellipse; which gives, eliminating y by the substitution $-\frac{b^2 x'}{a^2 y'}\, x$,

$$\frac{b^4}{a^2} \cdot \frac{x'^2}{y'^2} \cdot x^2 + b^2 x^2 = a^2 b^2,$$

$$\text{or } x^2 (b^2 x'^2 + a^2 y'^2) = a^4 y'^2,$$

$$\text{or } x^2 = \frac{a^2 y'^2}{b^2};$$

$$\therefore\ x = CM = -\frac{ay'}{b},$$

$$\text{and } y = DM = \frac{bx'}{a}.$$

143. The sum of the squares of any two semi-conjugate diameters is equal to the sum of the squares of the semi-axes.

$CP^2 = x'^2 + y'^2$,

$$CD^2 = \frac{a^2y'^2}{b^2} + \frac{b^2x'^2}{a^2} = a^2 - x'^2 + b^2 - y'^2, \text{ since } y'^2 = \frac{b^2}{a^2}(a^2 - x'^2);$$

$$\therefore\ CP^2 + CD^2 = a^2 + b^2.$$

144. All parallelograms whose sides touch an ellipse at the extremities of a pair of conjugate diameters, are equal to one another.

Draw PF (fig. 46) perpendicular to DC produced.

Then area of whole parallelogram

$$= 4CD.PF$$

$$= 4CD.CT.\sin TCF = 4CT.DM$$

$$= 4.\frac{a^2}{x'}.\frac{bx'}{a} = 4ab \text{ (Art. 123).}$$

145. If we denote CP, CD by a', b', and the angle DCP by γ, we have

$PF = a'\sin\gamma$, and therefore $a'b'\sin\gamma = CD.PF = ab$.

146. Also if we denote CQ or PF by p, we have

$$p^2 = \frac{a^2b^2}{CD^2} = \frac{a^2b^2}{a^2 + b^2 - a'^2},$$

a relation between the central distance of a point, and the perpendicular upon the tangent at that point, let fall from the center.

147. The rectangle contained by the focal distances of any point is equal to the square of the corresponding semi-conjugate diameter.

$$CD^2 = a^2 + b^2 - CP^2 \text{ (fig. 40)}$$
$$= a^2 + b^2 - x^2 - b^2 + \frac{b^2}{a^2} x^2$$
$$= a^2 - e^2 x^2 = (a + ex) . (a - ex)$$
$$= SP . HP \text{ (Art. 115)}.$$

148. To find the equation to the ellipse referred to any pair of conjugate diameters as axes.

The equation to the ellipse referred to its center and axes is

$$a^2 y^2 + b^2 x^2 = a^2 b^2.$$

Let the conjugate diameters CP, CD (fig. 47), be the new axes of x' and y', inclined to the axis of x at angles $PCA = \alpha$, $DCA = \beta$; then since the origin remains unaltered, the formulæ for passing from the rectangular to the oblique axes will be (Art. 42),

$$x = x' \cos \alpha + y' \cos \beta, \quad y = x' \sin \alpha + y' \sin \beta.$$

Hence, substituting and reducing,

$$(a^2 \sin^2 \alpha + b^2 \cos^2 \alpha) x'^2 + (a^2 \sin^2 \beta + b^2 \cos^2 \beta) y'^2$$
$$+ 2 (a^2 \sin \alpha \sin \beta + b^2 \cos \alpha \cos \beta) x'y' = a^2 b^2.$$

$$\text{But } \tan \alpha \tan \beta = -\frac{b^2}{a^2} \text{ (Art. 140)};$$

$$\therefore \; a^2 \sin \alpha \sin \beta + b^2 \cos \alpha \cos \beta = 0.$$

Also if $CP = a'$, $CD = b'$, we have (Art. 120),

$$a'^2 (a^2 \sin^2 \alpha + b^2 \cos^2 \alpha) = a^2 b^2,$$
$$b'^2 (a^2 \sin^2 \beta + b^2 \cos^2 \beta) = a^2 b^2;$$

therefore, substituting and dividing by $a^2 b^2$, we get for the required equation,

$$\frac{x'^2}{a'^2} + \frac{y'^2}{b'^2} = 1,$$

or, in a geometrical form, supposing PC produced to meet the ellipse in G,

$$QV^2 = \frac{CD^2}{CP^2} . PV . VG.$$

149. This equation, which, suppressing the accents of the variables, is $\frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 1$, being precisely of the same form as that relative to the axes, it follows that all properties which do not depend upon the inclination of the co-ordinates, will be common to the axes of the ellipse and to its conjugate diameters.

Hence, x, y being the co-ordinates of any point Q referred to the conjugate diameters CP, CD (fig. 47), the equation to the tangent at that point will be

$$a'^2yy' + b'^2xx' = a'^2b'^2,$$

and if it meet the axis of x in T, we shall have

$$CT = \frac{CP^2}{CV} \text{ as before.}$$

150. Also if we wish to draw a tangent through an external point Q (fig. 48), whose co-ordinates are h and k, we shall have, to find the points of contact, the equations

$$a'^2y'^2 + b'^2x'^2 = a'^2b'^2,$$

$$a'^2ky' + b'^2hx' = a'^2b'^2.$$

And those points, as in preceding similar cases, may be determined by constructing the line whose equation is

$$a'^2ky + b'^2hx = a'^2b'^2,$$

that is, by taking $CT = \frac{a'^2}{h}$, $CR = \frac{b'^2}{k}$, and joining RT which will cut the ellipse in the two required points.

Since the distance CT is independent of k, if through Q we draw a line parallel to CD, and from any other point in this line we draw a pair of tangents to the ellipse, the secant passing through the new points of contact, will cut the diameter CP in T, as this point can only alter when h alters. Hence if from the several points of any straight line pairs of tangents be drawn to an ellipse, the straight lines which join the corresponding points of contact will all pass through the same point.

151. The tangents at the extremities of any chord will intersect in the diameter of which the chord is an ordinate.

For taking that diameter and its conjugate as the axes of x and y, the equation to the tangent will be

$$b'^2 x x' \pm a'^2 y y' = a'^2 b'^2,$$

where the upper or lower sign is to be used according as we consider the point Q (x', y'), or the other extremity of the chord Q' whose co-ordinates are x', $-y'$; and in both cases when $y = 0$, $x = \dfrac{a'^2}{x'}$, therefore the tangents meet the axis of x in the same point T, (fig. 47).

152. If from any point within or without an ellipse, two lines be drawn parallel to two given straight lines to meet the curve, the rectangle of the segments will be to one another in an invariable ratio.

Let O (fig. 49) be the given point with co-ordinates h and k. Then the polar equation to the ellipse, reckoning from O, will be (Art. 44)

$$a^2 (r \sin\theta + k)^2 + b^2 (r \cos\theta + h)^2 = a^2 b^2,$$

which is of the form $r^2 + Mr - N = 0$,

where $N = \dfrac{a^2 b^2 - a^2 k^2 - b^2 h^2}{a^2 \sin^2\theta + b^2 \cos^2\theta} = r' r''$, if these be the two values of r.

Now let Pp, Qq be drawn parallel to CP', CQ', which make given angles α, β with Cx, then

$$PO \times Op : QO \times Oq :: a^2 \sin^2\beta + b^2 \cos^2\beta : a^2 \sin^2\alpha + b^2 \cos^2\alpha$$
$$:: CP'^2 : CQ'^2 \text{ (Art. 120)},$$

a ratio independent of the position of the point O.

153. To find the area of the ellipse.

Let $APQRTA'$ (fig. 50) be any polygon inscribed in the ellipse, and let the ordinates PN, QM, &c. be produced to meet the circle on the major axis in p, q, r, &c. and join Ap, pq, &c.

Then area of trapezium $PNMQ = \frac{1}{2}(PN + QM) \,.\, NM$

$$= \frac{1}{2}\frac{b}{a}(pN + qM) \,.\, NM$$

$$= \frac{b}{a} \,.\, \text{trapezium } pNMq,$$

$$\text{or } \frac{\text{area of trapezium } PM}{\text{area of trapezium } pM} = \frac{b}{a},$$

and since the same ratio exists between every two corresponding trapeziums,

$$\frac{\text{area of polygon } APQA'}{\text{area of polygon } ApqA'} = \frac{b}{a},$$

and this is true however much the number of the sides of the polygons be increased; therefore supposing the number to be infinite, in which case the ratio of the polygons becomes that of the semi-ellipse and semicircle,

$$\frac{\text{area of semi-ellipse}}{\text{area of semicircle}} = \frac{b}{a};$$

$$\therefore \text{ area of ellipse} = \frac{b}{a}\pi a^2 = \pi a b.$$

154. Let K (fig. 51) be any point in the axis, and QPN an ordinate to the circle and ellipse, then

$$\text{elliptic area } ANP = \frac{b}{a} \cdot \text{circular area } ANQ,$$

$$\text{and triangle } PKN = \frac{b}{a} \cdot \text{triangle } QKN;$$

therefore, subtracting,

$$\text{the elliptic sectorial area } AKP = \frac{b}{a} \cdot \text{circular area } AKQ.$$

Also if a', b' be two semi-conjugate diameters and γ the angle between them, the area of the ellipse $= \pi a' b' \sin \gamma$ (Art. 145).

And the area of the sector bounded by the semi-diameters

$$= \tfrac{1}{4} \pi a' b' \sin \gamma.$$

SECTION VIII.

ON THE HYPERBOLA.

Various forms of the equation to the Hyperbola.

155. To find the equation to the Hyperbola.

The hyperbola is the locus of a point whose distance from a given point is always greater than its distance from a given fixed line in a constant ratio.

Let KK' (fig. 52) be the given fixed line, and S the given point, from which draw SX perpendicular to KK'. Let P be a point in the hyperbola on either side of KK', and from P draw PM perpendicular to KK' and join SP, and let the constant ratio of SP to PM be $e : 1$, e being greater than 1. Divide the given distance SX in A so that $SA = e \,.\, AX$, then A is a point in the curve, and assuming $AS = p$, we have $AX = \dfrac{p}{e}$. Through A draw Ay parallel to KK' and take A for the origin and Ax, Ay, for the co-ordinate axes, and let $AN = x$, $NP = y$ be the co-ordinates of P; then

$$SP^2 = e^2 \,.\, PM^2,$$

$$\text{or } SN^2 + NP^2 = e^2 \,.\, NX^2,$$

$$\text{or } (x-p)^2 + y^2 = e^2 \,.\left(\frac{p}{e} + x\right)^2 = (p + ex)^2;$$

$$\therefore\ y^2 = 2p\,(e+1)\,x + (e^2-1)\,x^2,$$

$$\text{or } y^2 = (e^2-1)\left(\frac{2p}{e-1}\,x + x^2\right),$$

or, if we replace the known quantity $\dfrac{p}{e-1}$ by a,

$$y^2 = (e^2-1)\,(2ax + x^2),$$

the required equation.

156. To determine the points where the curve cuts the axis of x, make $y=0$, then $x=0$, or $x=-2a$; the value $x=0$ gives the point A already known, the other value $x=-2a=AA'$, determines the point A'. Bisect AA' in C, then in the equation to the hyperbola making $x=-a=AC$, we get $y^2=-(e^2-1)a^2$ or $y=\pm a\sqrt{e^2-1}\,.\sqrt{-1}$, which are imaginary values; hence the curve does not, as in the case of the ellipse, meet the line BB' drawn perpendicular to AA' through its middle point; if however we put $a\sqrt{e^2-1}=b$ and take BC, $B'C$, each equal to b, BB' will be denoted by $2b$, and the equation will become

$$y=\pm\frac{b}{a}\sqrt{2ax+x^2}.$$

157. In order to transfer the origin to C, we must change x into $x'-a$, since $AN=CN-CA$;

$$\therefore\ y^2=\frac{b^2}{a^2}\left\{2a(x'-a)+(x'-a)^2\right\}=\frac{b^2}{a^2}(x'^2-a^2).$$

This form of the equation shews that the origin is the center of the hyperbola, and that the co-ordinate axes are axes of the hyperbola (Art. 64); but the term axis is more particularly appropriated to the portions of those lines, $AA'=2a$, $BB'=2b$; the former of which meets the hyperbola and is called the transverse axis, and its extremities are called the vertices of the hyperbola; and the latter, although the line along which it is measured does not meet the curve, is taken for the second axis of the hyperbola and is called the conjugate axis. Since $b=a\sqrt{e^2-1}$ where e is only restricted to be greater than 1, b may be either greater or less than a.

158. To trace the hyperbola by means of its equation.

The equation to the hyperbola referred to its axes is

$$y=\pm\frac{b}{a}\sqrt{x^2-a^2};$$

hence for all values of x between $+a$ and $-a$, y is imaginary, and therefore no part of the curve lies in the space

bounded by the two indefinite lines through A, A', parallel to BC. When $x = a$, $y = 0$, and as x increases positively from a to ∞, the two values of y are real and increase from zero to ∞, and give the infinite branch ZAz situated symmetrically with respect to Cx; and since when the sign of x is changed the values of y do not alter, the negative values of x will give a branch $Z'A'z'$ precisely similar to the former on the other side of BB', which is described by taking $SP' : P'M$ as $e : 1$. Moreover the two opposite branches of which the hyperbola is composed will everywhere turn their convexities towards the axis BB', otherwise a straight line might intersect them in more than two points, which is impossible.

159. Since the hyperbola is symmetrically situated with respect to its axes, if we take $CH = CS$, $CX' = CX$, and draw kX' parallel to CB, the curve may be described by means of the focus H and directrix kX', exactly in the same way as by means of S and KX. Hence the hyperbola has two foci situated in the transverse axis at equal distances from its center. Also since

$$a = \frac{p}{e-1} \text{ we have } AS = a(e-1).$$

$$\therefore \quad SC = a + a(e-1) = ae, \text{ and } e = \frac{SC}{AC}.$$

160. The quantity e which expresses the ratio of the distance between either focus and the center to the semi-transverse axis is called the eccentricity. Since $b = a\sqrt{e^2-1}$, the eccentricity expressed in terms of the semi-axes is equal to $\dfrac{\sqrt{a^2+b^2}}{a}$. Hence $SC = \sqrt{a^2+b^2}$, and $AS \, . \, AH = b^2$.

161. Since $AS = e \, . \, AX$, we have $AX = \dfrac{a(e-1)}{e}$.

$$\therefore \quad CX = \frac{a}{e} = \frac{a^2}{ae} = \frac{CA^2}{CS}.$$

which determines the directrix relative to the center.

162. The double ordinate passing through the focus is called the latus rectum. To find its value make $x = CS = ae$ (fig. 53).

$$\therefore \quad y^2 = \frac{b^2}{a^2}(a^2e^2 - a^2) = b^2(e^2-1) = \frac{b^4}{a^2};$$

$$y = \pm\frac{b^2}{a}, \quad \therefore \quad LL' = \frac{2b^2}{a} = 2a(e^2-1).$$

163. In the hyperbola when the distance CS between the focus and center becomes infinite, the distance AS between the focus and vertex remaining finite, the curve is changed into a parabola.

The equation is $y^2 = 2a(e^2-1)x + (e^2-1)x^2$,

or $y^2 = 2p(e+1)x + (e^2-1)x^2$ if $AS = p$.

But $e = \dfrac{SC}{AC} = \dfrac{SC}{SC-p} = \dfrac{1}{1-\dfrac{p}{SC}}$; let $SC = \infty$, $\therefore e = 1$;

$\therefore \; y^2 = 4px$, the equation to a parabola.

164 When $b = a$, the above equations become

$$y^2 = x^2 - a^2, \quad y^2 = 2ax + x^2.$$

The hyperbola in this case is called rectangular, and it is to the ordinary hyperbola what the circle is to the ellipse.

165. Since $y^2 = \dfrac{b^2}{a^2}(x^2 - a^2) = \dfrac{b^2}{a^2}(x+a)(x-a)$,

gives $PN^2 = \dfrac{BC^2}{AC^2}.A'N.AN$,

we see that the square of the ordinate varies as the rectangle of the distances of its foot from the extremities of the transverse axis.

166. The equation to the hyperbola results from that to the ellipse by changing b^2 into $-b^2$, or b into $b\sqrt{-1}$.

This remark may be of use in enabling us to foresee those properties of the hyperbola which are analogous to properties of the ellipse.

167. To express the distances of any point in the hyperbola from the foci in terms of its abscissa.

$$SP^2 = SN^2 + NP^2 \text{ (fig. 53)}$$
$$= (x - ae)^2 + (e^2 - 1)(x^2 - a^2)$$
$$= (ex - a)^2;$$

$\therefore$ $SP = \pm (ex - a)$; now as long as P is in that branch of the hyperbola of which S is the interior focus, ex is greater than a; therefore in order that SP may be positive we must take the upper sign, which gives

$$SP = ex - a.$$

Also $\quad HP^2 = (x + ae)^2 + (e^2 - 1)(x^2 - a^2) = (ex + a)^2.$

$$\therefore \quad HP = ex + a.$$

168. Exactly in the same way as for the ellipse, it may be shewn that the foci are the only points whose distances from any point in the curve can be expressed rationally in terms of the abscissa of the point. (Art. 116).

169. Hence, subtracting,

$$HP - SP = 2a,$$

or the difference of the focal distances of any point in the hyperbola is constant, and equal to the transverse axis. This property affords a simple method of determining any number of points in a hyperbola of which we know the transverse axis and foci. In $A'A$ (fig. 53) produced take any point F and with centers S and H and radii respectively equal to AF, $A'F$, describe circles intersecting in P, P'; these are manifestly points in the hyperbola. The curve may be described by a continuous motion if we have a rule HM (fig. 53) moveable about the focus H, and a

string SPM fastened to M and to the other focus S, of such a length that $HM - SPM = AA'$; then as HM revolves about H, if a point P slide along HM so as always to confine a portion of string PM against it, the point will trace out a hyperbola.

This property also furnishes the following method of investigating the equation to the hyperbola.

170. To find the locus of a point the difference of whose distances from two fixed points is constant.

Through the two fixed points S, H (fig. 53) draw the indefinite line Hx, bisect SH in C and through C draw Cy perpendicular to it, and take Cx, Cy for the axes of the co-ordinates, as the locus will evidently be symmetrical with respect to these lines. Let $SC = CH = c$, $CN = x$, $NP = y$, the co-ordinates of any point, and $HP - SP = 2a$.

$$\text{Then } HP^2 = (x+c)^2 + y^2,$$

$$SP^2 = (x-c)^2 + y^2;$$

$$\therefore\ HP^2 - SP^2 \quad \text{or} \quad 2a(HP + SP) = 4cx;$$

$$\therefore\ HP + SP = \frac{2cx}{a};$$

$$\text{but } HP - SP = 2a;$$

$$\therefore\ HP = \frac{cx}{a} + a;$$

$$\therefore\ \left(\frac{cx}{a} + a\right)^2 = (x+c)^2 + y^2,$$

$$\text{or } y^2 = \frac{c^2 - a^2}{a^2}(x^2 - a^2).$$

Now $HP - SP$ is less than SH or $a < c$, consequently $c^2 - a^2$ is a positive quantity, and the equation, as we should expect, represents a hyperbola whose transverse axis is $2a$ and conjugate axis $2\sqrt{c^2 - a^2}$.

171. To find the polar equation to the hyperbola, one of the foci being the pole.

Let the interior focus be the pole, $SP = r$, $xSP = \theta$ (fig. 53), then $SN = r\cos\theta$ and $CN = ae + r\cos\theta$;

$$\text{but } SP = e\,.\,CN - a;$$

$$\therefore\ r = ae^2 + er\cos\theta - a,$$

$$\text{or } r = \frac{a(e^2-1)}{1 - e\cos\theta}.$$

Since $e > 1$, there is some angle $xSD = \alpha$ whose cosine $= \frac{1}{e}$; for values of θ less than α, r is negative and there are no points in the branch AZ corresponding to those values, because $SP = ex - a$ is always positive. When $\theta = \alpha$, r is infinite and the radius vector meets the curve at an infinite distance; when θ exceeds α, r is positive, and as θ increases to π we get the portion of the curve ZA; when θ increases beyond π, the same values of r recur in an inverse order, giving the portion Ax till $\theta = 2\pi - \alpha$, when r is again infinite and afterwards becomes negative.

172. If S be the exterior focus and $SP' = r$, $xSP' = \theta$,

$SP' = e.CN' + a$; and $CN' = SP'\cos P'SN' - ae = -r\cos\theta - ae$;

$$\therefore\ r = -re\cos\theta - ae^2 + a,$$

$$\text{or } r = \frac{-a(e^2-1)}{1 + e\cos\theta}.$$

In this case also, r is negative till $\theta = \pi - \alpha$, when it becomes infinite, and then produces the branch of the hyperbola $Z'A'x'$ as θ changes from $\pi - \alpha$ to $\pi + \alpha$.

There is no difficulty in shewing that if we remove the restriction of having r positive, and measure negative values upon the radius vector produced backwards, the same equation will represent both branches of the hyperbola.

$$y^2 = \frac{b^2}{a^2}(x^2 - a^2)$$

$$a^2 y^2 = b^2 x^2 - b^2 a^2$$

$$a^2 y^2 - b^2 x^2 = -b^2 a^2$$

173. To find the polar equation to the hyperbola, the center being the pole.

Let $CP = r$, $xCP = \theta$ (fig. 53), then $x = r\cos\theta$, $y = r\sin\theta$, and substituting in the equation $a^2y^2 - b^2x^2 = -a^2b^2$, we get

$$r^2(a^2\sin^2\theta - b^2\cos^2\theta) = -a^2b^2,$$

or dividing by a^2 and observing that $\frac{b^2}{a^2} = e^2 - 1$,

$$r^2(1 - e^2\cos^2\theta) = -b^2;$$

$$\therefore\ r = \frac{b}{\sqrt{e^2\cos^2\theta - 1}}.$$

Taking θ from zero to α, where $\cos\alpha = \frac{1}{e}$, we get the part AZ; from $\theta = \alpha$ to $\theta = \pi - \alpha$, r is imaginary; from $\theta = \pi - \alpha$ to $\theta = \pi + \alpha$ we get the branch $Z'A'z'$; and the remaining portion Az in taking θ from $2\pi - \alpha$ to 2π.

Tangent and Normal to the Hyperbola.

174. To find the equation to the tangent of a hyperbola at a given point.

The equation to the hyperbola being $a^2y^2 - b^2x^2 = -a^2b^2$, in order to find the equation to the tangent the process will be the same as for the ellipse, with the sole difference that b^2 will every where be replaced by $-b^2$, and the result will be the same, subject to that modification. Hence if x', y', be the co-ordinates of the point of contact and α the angle which the touching line makes with the axis of x,

$$\tan\alpha = \frac{b^2x'}{a^2y'}, \quad \text{(Art. 121)}$$

and the equation to the tangent will be

$$a^2yy' - b^2xx' = -a^2b^2.$$

175. The formula $\tan\alpha = \frac{b^2}{a^2}\frac{x'}{y'}$,

since it does not alter when x' and y' are replaced by $-x'$ and $-y'$, shews that if PC (fig. 54) be produced to meet the hyperbola in P', the tangents at P and P' are parallel as we might have foreseen on account of the symmetrical position of the hyperbola relative to its axes; and that at the points A, A' for which $y'=0$, $x'=\pm a$, $\tan\alpha=\infty$, or the tangents are perpendicular to the axis. Also replacing x' by its value

$$\frac{a}{b}\sqrt{y'^2+b^2}, \text{ we get } \tan\alpha = \frac{b}{a}\sqrt{1+\frac{b^2}{y'^2}},$$

which shews that as y' increases from zero to ∞, $\tan\alpha$ diminishes from ∞ to a limit $\frac{b}{a}$.

176. To find where the tangent meets the transverse axis, make in the equation to the tangent $y=0$;

$$\therefore\ x=\frac{a^2}{x'}, \text{ or } CT=\frac{CA^2}{CN}.$$

As this result is independent of b, it will be the same for all hyperbolas described with the same transverse axis. The value of CT diminishes as x' increases, but is always of the same sign, and becomes zero when $x'=\infty$. Hence when the point of contact is at an infinite distance, the tangent passes through the center and makes with the axis of x an angle whose tangent is $\frac{b}{a}$; i. e. it coincides with the diagonal CW of the rectangle constructed with the semi-axes. This line is called an asymptote to the hyperbola.

177. The subtangent $NT=CN-CT=x'-\frac{a^2}{x'}=\frac{x'^2-a'^2}{x'}$.

178. To find the equation to the normal of a hyperbola at a given point.

Let x', y', be the co-ordinates of the given point, then since the normal is perpendicular to the tangent at that point, it is easily seen that its equation is

$$y - y' = -\frac{a^2 y'}{b^2 x'}(x - x').$$

179. To determine the point G (fig. 54) where the normal meets the transverse axis; make in the equation to the normal

$$y = 0; \quad \therefore \; x = x'\left(1 + \frac{b^2}{a^2}\right), \text{ or } CG = e^2 . CN.$$

Hence, the least value of CG, when $x' = a$, is $\dfrac{a^2 + b^2}{a}$.

The subnormal $NG = CG - CN = \dfrac{b^2}{a^2} . x' = \dfrac{b^2}{a^2} . CN.$

180. The normal at any point bisects the exterior angle formed by the focal distances of that point.

For $SG = CG - CS = e^2 x' - ae = e . SP$,

$HG = CG + CH = e^2 x' + ae = e . HP$;

$\therefore \dfrac{SG}{HG} = \dfrac{SP}{HP}$; $\therefore$ PG bisects the angle SPh. (Euc. VI. Prop. A.)

Also since GPT, GPt, are right angles, and $SPG = hPG$;

$$\therefore \; SPT = hPt = HPT,$$

or the focal distances make equal angles with the tangent on opposite sides of it.

181. These properties furnish a simple method of drawing a tangent to the hyperbola through a given point.

First, let the point be in the curve as P (fig. 55), join SP, HP; make $HK = 2AC$, join SK and draw PY perpendicular to it, then $PK = HP - 2AC = SP$, PY is common, and the angles at Y right angles; $\therefore$ $\angle SPY = KPY$, and consequently PY is a tangent at P.

182. Next let the point be without the hyperbola, as T (fig. 56).

With center H and radius $= 2AC$ describe a circle KK', and with center T the given external point and radius TS describe a circle cutting the former in K, K', join HK meeting the hyperbola in P, and join TP, then TP is a tangent at P.

for $SP = HP - 2AC = PK$, $TS = TK$, and PT is common,

$\therefore$ PT bisects the angle SPH and is therefore a tangent at P.

Similarly, if HK' be joined and produced to meet the hyperbola in P', a second point of contact will be determined.

183. The locus of the feet of the perpendiculars dropped from the foci upon the tangent to a hyperbola is the circumference of the circle whose diameter is the transverse axis.

For joining CY (fig. 55), since SH is bisected in C, and SK in Y, CY is parallel to HK and $= \frac{1}{2}HK = AC$. Also drawing HZ, CQ perpendicular to ZY, Q is the middle point of ZY, and therefore $CZ = CY = CA$.

184. Since C is the center of the circle which is the locus of Y and Z, and SYZ is a right angle, if SY and ZC be produced to meet in S', this will be a point in the circumference and $SS' = HZ$.

$$\therefore \quad SY \times HZ = SY \times SS' = SA \times SA' = BC^2. \text{ (Art. 160)}$$

185. Since $SY \times HZ = BC^2$, and $\dfrac{SY}{SP} = \dfrac{HZ}{HP}$ or $\dfrac{SY}{HZ} = \dfrac{SP}{HP}$,

$$\therefore \quad SY^2 = BC^2 \times \frac{SP}{HP};$$

or if SP, SY, be denoted respectively by r and p, and consequently $HP = 2a + r$, we have

$$p^2 = b^2 \cdot \frac{r}{2a + r}.$$

186. Draw CE parallel to PY (fig. 55), then CP is a parallelogram, and $PE = CY = CA$.

The Hyperbola referred to its Conjugate Diameters.

187. To find the locus of the middle points of a system of parallel chords.

Let the chords be parallel to a line CW through the center (fig. 57), whose equation is $y = mx$; then the equation to any one of the chords QQ' will be $y = mx + c$, and to determine the points in which it meets the hyperbola, we must combine its equation with that to the hyperbola.

$$a^2y^2 - b^2x^2 = -a^2b^2;$$

This gives, eliminating x by the substitution $\frac{1}{m}(y - c)$,

$$y^2 + \frac{2b^2c}{m^2a^2 - b^2} \cdot y + b^2 \cdot \frac{m^2a^2 - c^2}{m^2a^2 - b^2} = 0,$$

the roots of which are represented by QM, $Q'M'$; but if V be the middle point of QQ' and $CN = X$, $NV = Y$, its co-ordinates, $2NV = QM + Q'M'$,

$$\therefore \quad Y = \frac{-b^2c}{m^2a^2 - b^2},$$

$$\text{and } X = \frac{1}{m}(Y - c) = -\frac{ma^2c}{m^2a^2 - b^2}.$$

Dividing one result by the other in order to eliminate the quantity c which particularizes the chord, we get

$$Y = \frac{b^2}{ma^2}X,$$

a relation between the co-ordinates of the middle point of any chord, and therefore the equation to its locus, which is consequently a straight line CV passing through the origin.

The straight line which passes through the middle points of a system of parallel chords is called a diameter; hence all diameters of a hyperbola pass through its center; and, conversely, every line through the center may be considered as a diameter.

188. Hence, denoting the equation to any chord by $y = mx + c$, and the equation to the diameter which bisects it by $y = m'x$, we have

$$m' = \frac{b^2}{ma^2}, \text{ or } mm' = \frac{b^2}{a^2},$$

a simple relation by means of which the equation of one may be deduced from that of the other.

189. The chords which are parallel to CV are bisected by CW.

For any one of these chords RR' may be represented by the equation

$$y = m'x + c',$$

then the diameter which bisects them will have for its equation

$$y = \frac{b^2}{m'a^2}x, \text{ or } y = mx,$$

which belongs to CW. Hence two diameters, whose equations $y = mx$, $y = m'x$ are so related that $mm' = \frac{b^2}{a^2}$, have the property that each bisects the chords parallel to the other; they are called conjugate diameters. But the term is usually restricted to those portions of them PP', DD', which are intercepted by the proposed hyperbola, and the hyperbola which is conjugate to it (fig. 58).

190. The latter is a hyperbola $BDB'D'$ whose transverse and conjugate axes coincide respectively with the conjugate and transverse axes of the proposed curve; and the employment of it is attended with great conveniences in stating and investigating the properties of the hyperbola.

The equation to the conjugate hyperbola, referred to the same axis of x and axis of y as the primitive hyperbola so that $DM = y$, $CM = x$, (fig. 58) will consequently be

$$x^2 = \frac{a^2}{b^2}(y^2 - b^2), \text{ or } y^2 = \frac{b^2}{a^2}(x^2 + a^2),$$

which we observe results from the equation to the primitive hyperbola by replacing a^2 and b^2 by $-a^2$ and $-b^2$.

191. If PT be a tangent at P (fig. 58), and x', y', the co-ordinates of P, then

$$m = \frac{y'}{x'}, \text{ and the equation to } PT \text{ is}$$

$$y - y' = \frac{b^2}{ma^2}(x - x')$$

which represents a line parallel to CD. Hence the tangent applied at the extremity of any diameter is parallel to the corresponding conjugate diameter. Similarly, the tangent to the conjugate hyperbola at D is parallel to CP; and if tangents be applied to the hyperbola and its conjugate, at the extremities of a pair of conjugate diameters, they will form a parallelogram.

192. Of any two conjugate diameters, only one can meet the hyperbola.

Let $y = mx$ be the equation to a diameter; to determine its intersection with the curve, put mx for y in the equation

$$a^2y^2 - b^2x^2 = -a^2b^2,$$

and we find for the abscissæ of the points of intersection

$$x = \pm\sqrt{\frac{a^2b^2}{b^2 - m^2a^2}},$$

which are real as long as m is less than $\frac{b}{a}$, but imaginary if m be greater than $\frac{b}{a}$; in the former case the diameter intersects the curve, in the latter it does not. But the relation $mm' = \frac{b^2}{a^2}$ shews that if m be less than $\frac{b}{a}$, m' is greater than $\frac{b}{a}$; hence every diameter which meets the hyperbola, has its conjugate diameter amongst those which do not meet it.

193. If we construct on the axes of the curve the rectangle Ll' (fig. 59), all the diameters which fall within the angle LCl, make with AC an angle whose tangent (abstracting the sign) is less than $\frac{b}{a}$; whilst the diameters which fall within the angle LCL' make with AC an angle whose tangent exceeds $\frac{b}{a}$; the former are those that meet the curve, the latter those which do not. In the particular case when $m = \frac{b}{a}$, we have also $m' = \frac{b}{a}$, and the conjugate diameters coincide with Ll', and meet the curve only at an infinite distance; similarly, when $m = -\frac{b}{a}$, we have $m' = -\frac{b}{a}$, and the two diameters coincide with the other diagonal $L'l$.

194. Having given the co-ordinates of the extremity of any diameter, to find those of the diameter conjugate to it.

Let CD be conjugate to CP (fig. 58), and let it meet the conjugate hyperbola in D; let x', y' be co-ordinates of P, and consequently $y = \frac{y'}{x'}x$ the equation to CP, then $y = \frac{b^2}{a^2}\cdot\frac{x'}{y'}x$ is the equation to CD; and to determine the co-ordinates of D we must combine this equation with the equation to the conjugate hyperbola, which is

$$a^2y^2 - b^2x^2 = a^2b^2.$$

This gives, eliminating y by the substitution of $\frac{b^2x'}{a^2y'}x$,

$$\frac{b^4}{a^2}\cdot\frac{x'^2}{y'^2}x^2 - b^2x^2 = a^2b^2,$$

$$\text{or } x^2(b^2x'^2 - a^2y'^2) = a^4y'^2,$$

$$\text{or } x^2 = \frac{a^2y'^2}{b^2};$$

$$\therefore\ x = CM = \frac{ay'}{b}, \quad \text{and } y = DM = \frac{bx'}{a}.$$

195. The difference of the squares of any two semi-conjugate diameters, is equal to the difference of the squares of the semi-axes.

$$CP^2 = x'^2 + y'^2,$$

$$CD^2 = \frac{a^2y'^2}{b^2} + \frac{b^2x'^2}{a^2} = x'^2 - a^2 + y'^2 + b^2, \text{ because } y'^2 = \frac{b^2}{a^2}(x'^2 - a^2);$$

$$\therefore\ CP^2 - CD^2 = a^2 - b^2.$$

196. All parallelograms whose sides touch a hyperbola and the conjugate hyperbola at the extremities of a pair of conjugate diameters, are equal to one another.

Draw PF perpendicular to CD (fig. 58), then

$$\text{area of whole parallelogram} = 4\,CD\,.\,PF = 4\,CD\,.\,CT \sin TCF$$

$$= 4\,DM\,.\,CT = 4\frac{bx'}{a}\,.\,\frac{a^2}{x'} = 4ab.$$

197. If we denote CP, CD by a', b', and angle PCD by γ, we have

$$PF = a' \sin\gamma, \quad \text{and} \;\therefore\; a'b' \sin\gamma = CD \times PF = ab.$$

Also if we denote PF by p, we have the relation between the central distance of any point and the perpendicular from the center upon the tangent at that point,

$$p^2 = \frac{a^2b^2}{CD^2} = \frac{a^2b^2}{a'^2 - a^2 + b^2}.$$

198. Draw the diagonal CL (fig. 58), which will pass through the middle point of DP, whose co-ordinates are equal to $\frac{1}{2}(CM + CN)$, $\frac{1}{2}(DM + PN)$;

$$\therefore\ \tan LCA = \frac{\frac{1}{2}\left(y' + \frac{bx'}{a}\right)}{\frac{1}{2}\left(x' + \frac{ay'}{b}\right)} = \frac{b}{a};$$

hence the diagonals of all the parallelograms coincide with the limiting positions of the tangents (Art. 176); and the parallelo-

grams are not only equal in area, but they all have their diagonals in the same straight lines.

199. The rectangle contained by the focal distances of any point, is equal to the square of the corresponding semi-conjugate diameter.

$$\begin{aligned} CD^2 &= CP^2 - a^2 + b^2 \\ &= x^2 + \frac{b^2}{a^2}x^2 - b^2 - a^2 + b^2 \\ &= e^2x^2 - a^2 = (ex + a) . (ex - a) \\ &= SP . HP. \end{aligned}$$

200. To find the equation to the hyperbola referred to any system of conjugate diameters as axes.

The equation to a hyperbola referred to its center and axes is

$$a^2y^2 - b^2x^2 = -a^2b^2.$$

Let the conjugate diameters CP, CD (fig. 60) be the new axes of x' and y' inclined to the axis of x at angles $PCA = \alpha$, $DCA = \beta$; then since the origin remains unaltered, the formulæ for passing from the rectangular to the oblique axes will be (Art. 42)

$$x = x' \cos\alpha + y' \cos\beta, \quad y = x' \sin\alpha + y' \sin\beta.$$

Hence, substituting and reducing,

$$\begin{aligned} (a^2 \sin^2\alpha - b^2 \cos^2\alpha)x'^2 + (a^2 \sin^2\beta - b^2 \cos^2\beta)y'^2 \\ + 2x'y(a^2 \sin\alpha \times \sin\beta - b^2 \cos\alpha \times \cos\beta) = -a^2b^2; \end{aligned}$$

but $\tan\alpha . \tan\beta = \dfrac{b^2}{a^2}$; therefore $a^2\sin\alpha . \sin\beta - b^2\cos\alpha . \cos\beta = 0$.

Also if $CP = a'$, $CD = b'$, we have (Art. 173)

$$a'^2 (a^2 \sin^2\alpha - b^2 \cos^2\alpha) = -a^2b^2,$$

$$b'^2 (a^2 \sin^2\beta - b^2 \cos^2\beta) = +a^2b^2;$$

hence substituting and dividing by $-a^2b^2$, we get for the required equation

$$\frac{x'^2}{a'^2} - \frac{y'^2}{b'^2} = 1,$$

or, in a geometrical form,

$$QV^2 = \frac{CD^2}{CP^2} PV \,.\, VG.$$

201. This equation, which, suppressing the accents of the variables, is

$$a'^2y^2 - b'^2x^2 = -a'^2b'^2,$$

being of precisely the same form as that relative to the axes, it follows that all properties which do not depend upon the inclination of the co-ordinates, will be common to the axes of the hyperbola and its conjugate diameters.

Hence the equation to the tangent at a point $Q(x', y')$ will be

$$a'^2yy' - b'^2xx' = -a'^2b'^2,$$

and if the tangent meet the axis of x in T, we shall have $CT = \dfrac{CP^2}{CV}$, as before; and if we wish to draw a tangent through an external point $Q(h, k)$ (fig. 61) we shall have, to determine the points of contact (x', y'), the equations

$$a'^2y'^2 - b'^2x'^2 = -a'^2b'^2,$$

$$a'^2ky' - b'^2hx' = -a'^2b'^2,$$

and if we construct the line represented by the latter, considering x' and y' as the variables, by taking $CT = \dfrac{a'^2}{h}$, $CR = \dfrac{-b'^2}{k}$ and joining RT, it will cut the hyperbola in the two points of contact.

202. Since the distance CT is independent of k, if through Q we draw a line parallel to CD, and from any

other point in this line we draw a pair of tangents to the hyperbola, the secant passing through the new points of contact will cut the diameter CP in T, as this point only changes when h changes. Hence from the several points of a straight line if pairs of tangents be drawn to a hyperbola, the straight lines which join the corresponding points of contact will all intersect in the same point.

203. The tangents at the extremities of any chord will intersect in the diameter of which the chord is an ordinate.

For taking that diameter and its conjugate as the axes of x and y, the equation to the tangent will be

$$\pm a'^2 yy' - b'^2 xx' = -a'^2 b'^2,$$

according as we consider the point Q (x', y'), or the other extremity of the chord Q' whose co-ordinates are x', $-y'$; and in both cases when $y = 0$, $x = \dfrac{a'^2}{x'}$, therefore the tangents meet the axis of x in the same point T. (fig. 60).

204. If from any point within or without a hyperbola, two lines be drawn parallel to two given straight lines to meet the curve, the rectangle of the segments will be to each other in an invariable ratio.

Let O (fig. 62) be the given point with co-ordinates h and k, then taking O for the pole, and measuring θ from a line parallel to the transverse axis, the equation to the hyperbola will be

$$a^2 (r \sin\theta + k)^2 - b^2 (r \cos\theta + h)^2 = -a^2 b^2,$$

which is of the form $r^2 + Mr - N = 0$,

$$\text{where } N = \frac{-a^2 b^2 - a^2 k^2 + b^2 h^2}{a^2 \sin^2\theta - b^2 \cos^2\theta} = r'r'',$$

if these be the two values of r.

Now let Pp, Qq, be drawn parallel to CP', CQ', which make angles α, β, with Cx, then

$$PO \times Op \;:\; QO \times Oq \;::\; a^2 \sin^2 \beta - b^2 \cos^2 \beta \;:\; a^2 \sin^2 \alpha - b^2 \cos^2 \alpha$$
$$:: \; CP'^2 \;:\; CQ'^2,$$

which ratio is independent of the position of the point O.

The Hyperbola referred to its Asymptotes.

205. The asymptotes of the hyperbola coincide with the diagonals of the parallelogram constructed upon any pair of conjugate diameters.

The equation to the diagonal CL (fig. 63) referred to the conjugate diameters CP, CD, is

$$y = \frac{b'}{a'} x.$$

And the equation to the hyperbola,

$$y = \frac{b'}{a'} \sqrt{x^2 - a'^2};$$

$$\therefore\; RQ = \frac{b'}{a'} (x - \sqrt{x^2 - a'^2}) = \frac{a'b'}{x + \sqrt{x^2 - a'^2}}.$$

Hence when $x = \infty$, RQ becomes zero; hence Ll' is an asymptote to the portions PQ, $P'Q'$; and similarly it may be shewn that $L'l$ is an asymptote to the other portions. Hence the asymptotes may be considered as the limits of the tangents (Art. 198).

206. If any chord of a hyperbola be produced to meet the asymptotes, the parts of it intercepted between the curve and the asymptotes will be equal.

Let Qq (fig. 63) any chord, when produced cut the asymptotes in R, r; bisect Qq in V, join CV, and refer the hyperbola to the diameter CP and its conjugate CD, then the equations to CR, Cr are

$$y = \frac{b'}{a'} x, \quad y = -\frac{b'}{a'} x;$$

$$\therefore\; VR = Vr, \text{ and } VQ = Vq; \;\therefore\; QR = qr.$$

207. If a line be drawn through P parallel to Rr, it will be a tangent at P, and $PL = Pl$.

Also $RQ \times Qr = RV^2 - QV^2 = \frac{b'^2}{a'^2}\{x^2 - (x^2 - a'^2)\} = b'^2 = CD^2$.

Hence when a line cutting the hyperbola is parallel to a diameter, the rectangle of the parts of this secant between a point of the curve and the asymptotes is equal to the square of the semi-diameter.

208. From any point P (fig. 63) of the hyperbola, draw parallels PG, PF to the asymptotes, and draw the tangent Ll which is bisected in P. Then the parallelogram GF is half of the triangle LCl.

But area LCl is constant, whatever be the position of P, and equals ab (Art. 196).

Hence, denoting by $2a$ the angle between the asymptotes, and by x, y, the co-ordinates of P referred to the asymptotes as axes, so that $CF = x$, $FP = y$, we get

$$xy \sin 2a = \frac{ab}{2}; \quad \text{but } \sin 2a = \frac{2 \tan a}{1 + \tan^2 a} = \frac{2ab}{a^2 + b^2};$$

$$\therefore \; xy = \tfrac{1}{4}(a^2 + b^2),$$

the equation to the hyperbola referred to its asymptotes.

209. To transform the equation to the hyperbola referred to its axes, into that representing the hyperbola referred to its asymptotes.

Let the inferior asymptote be the axis of x'; then since the axes of x' and y' make with the axis of x the angles $2\pi - a$ and a respectively, the formulæ are (Art. 42)

$$x = x' \cos a + y' \cos a, \quad y = y' \sin a - x' \sin a;$$

$$\therefore \; a^2 \sin^2 a \,(y' - x')^2 - b^2 \cos^2 a \,(y' + x')^2 = - a^2 b^2,$$

but $\tan\alpha = \frac{b}{a}$; and $\therefore\ a^2\sin^2\alpha = b^2\cos^2\alpha = \dfrac{b^2}{1+\dfrac{b^2}{a^2}} = \dfrac{a^2b^2}{a^2+b^2}$;

$$\therefore\ \frac{a^2b^2}{a^2+b^2}\cdot 4x'y' = a^2b^2, \text{ or } x'y' = \frac{a^2+b^2}{4}.$$

210. To find the equation to the line touching the hyperbola at a given point, when referred to its asymptotes.

Let the co-ordinates of the given point be x', y', and those of a point near it x'', y''; the equation to the line passing through them will be

$$y - y' = \frac{y''-y'}{x''-x'}(x-x'),$$

$$\text{but } x'y' = \tfrac{1}{4}(a^2+b^2),\quad x''y'' = \tfrac{1}{4}(a^2+b^2);$$

$$\therefore\ x''y'' - x'y' = 0, \text{ or } x'(y''-y') + y''(x''-x') = 0;$$

$$\therefore\ \frac{y''-y'}{x''-x'} = -\frac{y''}{x'}.$$

Hence the equation to the secant becomes

$$y - y' = -\frac{y''}{x'}(x-x'),$$

and in order that it may become a tangent we must suppose $y'' = y'$ which gives

$$y - y' = -\frac{y'}{x'}(x-x').$$

211. To find where it cuts the asymptotes, make $y = 0$; $\therefore\ x = 2x'$,

or $Cl = 2CF$, (fig. 63).

SECTION IX.

ON THE SECTIONS OF THE CONE AND CYLINDER.

---

212. THE surface described by an indefinite straight line which is carried round the perimeter of a given circle, always passing through a fixed point, is called a cone (fig. 64).

The circle is called the base of the cone, and the fixed point its vertex, and the line joining the vertex and center of the base is called the axis. The cone is moreover right or oblique according as the axis is at right angles, or inclined, to the plane of the base.

As the generating line is unlimited in both directions from the vertex, the surface of the cone is composed of two portions or sheets, perfectly similar, situated on opposite sides of the vertex. Also from the mode of generation it follows that every plane parallel to the base will cut the cone in a circle; and every plane through the axis will cut it in two straight lines. When the surface is a right cone, every generating line will make the same angle with the axis. The different curves obtained by cutting a cone by a plane are called Conic Sections.

213. All conic sections are curves of the second order.

Let PAP' (fig. 65) be a section of a right cone made by any plane, and through the axis VO draw a plane perpendicular to that of the section, cutting the cone in the lines VB, VD, and the plane of the section in the line AN, which take for the axis of x. Through any point P draw a plane perpendicular to the axis, intersecting the cone in the circle MPQ, and the plane of the section in PP';

then MQ will be a diameter of the circle, and PN will be at right angles to both AN and NM, and will consequently be a common ordinate to the circle and conic section AP. Draw Ay parallel to PN and take it for the axis of y, and let $AN = x$, $PN = y$; and choosing the data so as to embrace every case, let $AV = d$, $\angle VAN = \theta$, $AVQ = 2\alpha$. Then

$$y^2 = MN \times NQ.$$

$$\text{But } \frac{MN}{AN} = \frac{\sin\theta}{\cos\alpha}; \quad \therefore MN = \frac{x \sin\theta}{\cos\alpha}.$$

And drawing NF parallel to QV,

$$\frac{AF}{AN} = \frac{\sin(2\alpha + \theta)}{\cos\alpha}; \quad \therefore AF = \frac{x \sin(2\alpha + \theta)}{\cos\alpha};$$

$$\therefore NQ = 2d \,.\, \sin\alpha - \frac{x \,.\, \sin(2\alpha + \theta)}{\cos\alpha};$$

$$\therefore y^2 = \frac{2d \,.\, \sin\alpha \,.\, \sin\theta}{\cos\alpha} x - \frac{\sin\theta \,.\, \sin(2\alpha + \theta)}{\cos^2\alpha} x^2,$$

the equation to a curve of the second order; therefore every conic section is a curve of the second order; and it will be an ellipse, hyperbola, or parabola, according as the second term is negative, or positive, or zero.

Now the second term can only change its sign when $\sin(2\alpha + \theta)$ changes its sign. Hence the section will be an ellipse as long as $2\alpha + \theta$ is less than π, and therefore AN meets VQ produced, or the cutting plane meets only one sheet of the cone.

It will be a hyperbola when $2\alpha + \theta$ is greater than π, and therefore AN and VQ intersect when produced backwards, and the cutting plane meets both the sheets of the cone.

It will be a parabola when $2\alpha + \theta = \pi$, and therefore AN, VQ are parallel, or the cutting plane is parallel to a generating line of the cone.

214. To determine the axes of the conic section, we have, comparing the equation (supposing it to represent an ellipse, and therefore $\sin(2\alpha+\theta)$ to be positive)

$$y^2 = \frac{2d\sin\alpha\,.\,\sin\theta}{\cos\alpha}x - \frac{\sin\theta\,.\,\sin(2\alpha+\theta)}{\cos^2\alpha}x^2$$

with $y^2 = \dfrac{2b^2}{a}x - \dfrac{b^2}{a^2}x^2$,

the latus rectum or $\dfrac{2b^2}{a} = 2d\,.\tan\alpha\,.\sin\theta$,

$$\text{and } \frac{b^2}{a^2} = \frac{\sin\theta\,.\,\sin(2\alpha+\theta)}{\cos^2\alpha};$$

$$\therefore\ 2a = \frac{2d\sin\alpha\,.\,\cos\alpha}{\sin(2\alpha+\theta)};$$

$$\therefore\ 2b^2 = \frac{2d^2\sin^2\alpha\sin\theta}{\sin(2\alpha+\theta)} \quad\text{or}\quad b = d\,.\sin\alpha\sqrt{\frac{\sin\theta}{\sin(2\alpha+\theta)}}.$$

215. The minor axis may however be more conveniently expressed in the following manner.

From the extremities of the axis major let fall perpendiculars $AF=f$, $A'G=g$ (fig. 64), upon the axis of the cone; through C the middle point of AA' draw a plane parallel to the base, cutting the section in BB' which is its minor axis, and the cone in the circle MBQ, then

$$BC^2 = MC \times CQ = A'G \times AF = fg,$$

because MC, being parallel to DA', $= \frac{1}{2}DA' = A'G$,

and similarly $CQ = AF$.

Hence the distance of the foci of the elliptic section $= AD$;

for dropping the perpendicular AE, $A'E = f+g$;

$$\therefore\ AD^2 = 4a^2 + 4g^2 - 4g(f+g) = 4a^2 - 4fg = 4(a^2-b^2);$$

$$\therefore\ AD = 2\sqrt{a^2-b^2} = \text{distance of foci.}$$

216. If in that section of a cone through the axis which is perpendicular to the plane of an elliptic section, we describe circles touching the generating lines of the cone and the axis of the section, the points of contact with the axis will be the foci of the section.

For the distance of the foci $= A'D'$ (fig. 66).

$$\text{But } A'D' = A'U' - D'U' = A'S - AU$$
$$= AA' - 2AS$$
$$\therefore \quad AS = \tfrac{1}{2}(AA' - A'D');$$

therefore S is a focus. Similarly, H may be shewn to be the other focus.

Produce UU' to meet AA' produced in X,

then from the similar triangles AUX, ADA',

$$\frac{AX}{AA'} = \frac{AU}{AD} \quad \text{or} \quad \frac{AX}{2AC} = \frac{AS}{2SC};$$

$$\therefore \quad \frac{AX}{AC} = \frac{AS}{SC} \quad \text{or} \quad \frac{CX}{AC} = \frac{AC}{SC};$$

therefore X is the point where the directrix meets the axis (Art. 110). Similarly, X' is the point where the other directrix meets the axis.

217. When the section is a hyperbola the equation is

$$y^2 = \frac{2d\sin\theta \,.\, \sin\alpha}{\cos\alpha}\,x - \frac{\sin\theta \,.\, \sin(2\alpha + \theta)}{\cos^2\alpha}\,x^2,$$

where $\sin(2\alpha + \theta)$ is a negative quantity, and consequently the second term is positive, which compared with

$$y^2 = \frac{2b^2}{a}\,x + \frac{b^2}{a^2}\,x^2$$

will determine the axes, as in the case of the ellipse. When $d = 0$ the equation becomes

$$y^2 = -\frac{\sin\theta \,.\, \sin(2\alpha + \theta)}{\cos^2\alpha}\,x^2,$$

which represents two generating lines of the cone.

In this case also the semi-conjugate axis is a mean proportional between the perpendiculars dropped from the vertices of the hyperbola upon the axis of the cone.

218. When the section is a parabola, or $2a + \theta = \pi$, the equation is

$$y^2 = \frac{2d \sin\theta \,.\, \sin a}{\cos a} x = 4d \,.\, \sin^2 a \,.\, x, \text{ since } \sin\theta = \sin 2a.$$

219. We must now demonstrate the converse proposition, namely, that curves of the second order are conic sections.

Every curve of the second order is contained in the equation

$$y^2 = 4px + nx^2,$$

where $4p$ is the latus rectum, and n the square of the ratio of the axes, abstracting the sign. What we have to demonstrate is, that the quantities p, n, and a being given, we can assign real values of d and θ which shall render the above equation identical with

$$y^2 = \frac{2d \sin a \,.\, \sin\theta}{\cos a} x - \frac{\sin\theta \,.\, \sin(2a + \theta)}{\cos^2 a} x^2.$$

Equating the coefficients of x and x^2 in the two equations, we get

$$\frac{d \sin a \,.\, \sin\theta}{\cos a} = 2p, \quad \frac{\sin\theta \,.\, \sin(2a + \theta)}{\cos^2 a} = -n,$$

the former of which will give a real value of d when θ is real; the latter may be transformed into

$$\tfrac{1}{2}\left\{\cos 2a - \cos(2a + 2\theta)\right\} = -n\cos^2 a,$$

$$\text{or } \cos 2(a + \theta) = 2(1 + n)\cos^2 a - 1.$$

In the ellipse n is negative and less than 1; hence the preceding value of $\cos 2(a + \theta)$ lies between $+1$ and -1, and therefore θ is always real; consequently any given ellipse

may be regarded as a section of any proposed right cone whatever.

In the hyperbola n is positive and of any magnitude; if the above value of $\cos 2(\alpha+\theta)$ be negative, it will be evidently less than 1, and α will be real; but if it be positive, we must have

$$2(1+n)\cos^2\alpha - 1 \text{ less than } 1,$$

$$\text{and } \therefore \cos\alpha \text{ less than } \frac{1}{\sqrt{1+n}}, \quad \text{or than } \frac{a}{\sqrt{a^2+b^2}};$$

but if ω be the angle which the asymptote makes with the transverse axis, $\cos\omega = \dfrac{a}{\sqrt{a^2+b^2}}$; $\therefore \cos\alpha < \cos\omega$, $\therefore \alpha > \omega$; and therefore in order that a given hyperbola may be cut from a given cone, the vertical angle of the cone must be not less than the angle between the asymptotes.

In the parabola $n=0$, therefore $\sin\theta = 0$ or $\sin(2\alpha+\theta)=0$; the first is inadmissible, for it makes $p=0$; the second gives $2\alpha+\theta=\pi$, which will always furnish a real value for θ; hence a given parabola may be cut from any proposed cone.

220. To determine the curve which results from the intersection of a right cylinder by a plane.

Let APA' (fig. 67) be a section of a right cylinder, $AA'D$ a section of the cylinder through its axis, perpendicular to the plane of the section. Through any point P draw a plane perpendicular to the axis of the cylinder, intersecting it in a circle whose diameter is MQ, and the plane of the section in PP' which will be perpendicular to MQ, AA', and will be a common ordinate of the section and circle.

Let $AN=x$, $NP=y$, $AA'=2a$, $AD=2r$,

$$\text{then } y^2 = MN \,.\, NQ;$$

$$\text{but } \frac{MN}{AD} = \frac{AN}{AA'}, \quad \text{or } MN = \frac{r}{a}x,$$

$$\frac{NQ}{AD} = \frac{NA'}{AA'}, \quad \text{or } NQ = \frac{r}{a}(2a - x);$$

$$\therefore \; y^2 = \frac{r^2}{a^2}(2ax - x^2),$$

the equation to an ellipse.

221. In the same manner the nature of the sections of an oblique cone may be determined; but this, as well as the discussion of the sections of Conoids or figures generated by the revolution of conic sections about their axes, may be more conveniently deferred to Geometry of Three Dimensions. There is however one property of the oblique cone which admits of a simple demonstration, viz. that it may be cut by other planes besides those parallel to its base, so that the sections may be circles, and which we shall give here.

Let VBD (fig. 68) be the principal section of an oblique cone, that is, a section made by a plane through its axis perpendicular to its base; and let MPQ, APA' be two sections made by planes perpendicular to BVD, and of which the former is parallel to the base, and is therefore a circle with diameter MQ, and as PN is perpendicular to MQ, we have $PN^2 = MN.NQ$; and the latter will also be a circle if angle $AA'V = ABD$, for in that case the triangles AMN, $A'NQ$ are similar, and $\dfrac{NA'}{NQ} = \dfrac{MN}{NA}$;

$$\therefore \; A'N.NA = MN.QN = PN^2,$$

and as PN is perpendicular to AA', the section, which is called a subcontrary section, is a circle; its plane is perpendicular to the principal section of the cone, and makes the same angle with one of the generating lines of the cone which are in that plane, as the plane of the base does with the other.

SECTION X.

ON THE GENERAL EQUATION OF CURVES OF THE SECOND ORDER, AND ON CERTAIN GENERAL PROPERTIES OF ALGEBRAICAL CURVES.

---

222. We shall now proceed to the reduction of the general equation of the second degree

$$ay^2 + bxy + cx^2 + dy + ex + f = 0,$$

where we suppose the co-ordinates rectangular; for if they were oblique, by transforming them to rectangular co-ordinates we should obtain an equation of the same degree as the above, and which could not therefore be more general than the one we have assumed. We shall prove, as affirmed at Art. 62, that this equation by giving a proper position and direction to the origin and axes of the co-ordinates, can always be reduced to one of the forms,

$$Ay^2 + Bx^2 = C,$$

$$y^2 = Ax,$$

the co-ordinates being rectangular; and therefore can never represent any other curve than one of those discussed in the preceding Sections. The principle of the method is to change the system of co-ordinates, without giving any particular values to the quantities which determine the position of the new axes. By that means, indeterminate quantities are introduced into the transformed equation, to which such values can afterwards be assigned as will destroy certain of its terms. Instead of altering both the origin and direction of the co-ordinate axes at once, it is more convenient to effect these changes separately, in the following manner.

223. The general equation of the second order being

$$ay^2 + bxy + cx^2 + dy + ex + f = \phi(x, y) = 0,$$

in order to get rid of the terms in x and y, we must change the origin without altering the direction of the axes, by putting (Art. 39) $x = x' + h$, $y = y' + k$; this gives

$$ay'^2 + bx'y' + cx'^2 + (2ak + bh + d)y' + (2ch + bk + e)x'$$
$$+ ak^2 + bkh + ch^2 + dk + eh + f = 0,$$

and equating the coefficients of x' and y' to zero, we get

$$\left.\begin{aligned} 2ak + bh + d = 0 \\ 2ch + bk + e = 0 \end{aligned}\right\} \dots\dots\dots (1),$$

which give for the co-ordinates of the new origin, provided $b^2 - 4ac$ be different from zero, the single pair of determinate values

$$h = \frac{2ae - bd}{b^2 - 4ac}, \qquad k = \frac{2cd - be}{b^2 - 4ac}.$$

Hence the equation becomes, suppressing the accents,

$$ay^2 + bxy + cx^2 + \phi(h, k) = 0,$$

where $\phi(h, k) = f + \frac{1}{2}(dk + eh) = f + \dfrac{cd^2 - bed + ae^2}{b^2 - 4ac}$,

as appears by multiplying equations (1) by k and h respectively, and taking their sum; and since this equation remains unaltered when we change x and y into $-x$ and $-y$, the new origin is the center of the curve.

224. We must now get rid of the term involving the product of the co-ordinates xy, by changing the direction of the axes. For this purpose (Art. 40) put

$$x = x' \cos\theta - y' \sin\theta, \quad y = x' \sin\theta + y' \cos\theta;$$

$$\therefore\ a(x'^2 \sin^2\theta + 2x'y' \sin\theta\cos\theta + y'^2 \cos^2\theta)$$
$$+ b(x'^2 \sin\theta\cos\theta + x'y' \cos^2\theta - x'y' \sin^2\theta - y'^2 \cos\theta\sin\theta)$$
$$+ c(x'^2 \cos^2\theta - 2x'y' \cos\theta\sin\theta + y'^2 \sin^2\theta) + \phi(h, k) = 0,$$

$$\text{or} \quad Ay'^2 + Bx'^2 + \phi(h, k) = 0,$$

$$\text{where} \quad \left.\begin{aligned} A &= a\cos^2\theta - b\cos\theta\sin\theta + c\sin^2\theta \\ B &= a\sin^2\theta + b\cos\theta\sin\theta + c\cos^2\theta \end{aligned}\right\} \ldots\ldots\ldots (2),$$

and the coefficient of $x'y'$

$$= 2a\sin\theta\cos\theta + b(\cos^2\theta - \sin^2\theta) - 2c\cos\theta\sin\theta = 0,$$

which must give a real value for θ, in order that the term involving $x'y'$ may disappear;

$$\therefore\ (a - c)\sin 2\theta + b\cos 2\theta = 0,$$

$$\text{or} \quad \tan 2\theta = \frac{-b}{a - c}.$$

As the tangent of an angle may have any magnitude, it follows that this equation will always give real values for 2θ; and if we denote by 2α that value of 2θ which lies between zero and π, then the positive values of 2θ are

$$2\alpha, \quad \pi + 2\alpha, \quad 2\pi + 2\alpha, \quad 3\pi + 2\alpha, \quad \&c.;$$

consequently as θ lies between zero and 2π (Art. 41), there are four values of θ, viz.

$$\alpha, \quad \frac{\pi}{2} + \alpha, \quad \pi + \alpha, \quad \frac{3\pi}{2} + \alpha,$$

the two former of which determine two lines at right angles to one another, and the two latter determine the prolongations of these lines; so that if we take one of these lines for the axis of x', the other will be the axis of y'. Hence there exists one system of rectangular axes, and one only, proper to make the product of the co-ordinates $x'y'$ disappear from the transformed equation.

If however we have, at the same time, $b = 0$ and $a = c$, $\tan 2\theta$ becomes indeterminate, or rather the coefficient of $x'y'$ is identically zero; this proves that we may in that case take any two rectangular axes whatever, without introducing the product of the co-ordinates into the transformed equation; and agrees with (Art. 48), for the curve is then a circle.

225. We shall now proceed to the actual determination of the axes of the curve. Since

$$\tan 2\theta = \frac{-b}{a-c};$$

$$\therefore\ \cos 2\theta = \frac{a-c}{\sqrt{(a-c)^2+b^2}},$$

$$\sin 2\theta = \cos 2\theta \,.\, \tan 2\theta = \frac{-b}{\sqrt{(a-c)^2+b^2}},$$

in these expressions the radical may have either the sign $+$ or $-$, because we are at liberty to choose either of the new axes for the axis of x; but to avoid all ambiguity, we shall take the radical with a positive sign; then $\sin 2\theta$ will have a sign contrary to that of b.

Hence taking the sum and difference of equations (2), and substituting the above values of $\cos 2\theta$ and $\sin 2\theta$, we get

$$A + B = a + c,$$

$$A - B = (a-c)\cos 2\theta - b\sin 2\theta = \frac{(a-c)^2+b^2}{\sqrt{(a-c)^2+b^2}} = \sqrt{(a-c)^2+b^2};$$

$$\therefore\ A = \tfrac{1}{2}\left\{a + c + \sqrt{(a+c)^2+m}\right\},$$

$$B = \tfrac{1}{2}\left\{a + c - \sqrt{(a+c)^2+m}\right\},$$

putting $m = b^2 - 4ac$.

226. We have now two cases to consider, according as m is positive or negative.

First let m be negative, then A and B have the same sign; and supposing $\phi(h, k)$ to be of a contrary sign to A and B, and $= -C$, the equation is

$$Ay^2 + Bx^2 = C,$$

which represents an ellipse with semi-axes $\sqrt{\dfrac{C}{A}}$, $\sqrt{\dfrac{C}{B}}$. If $\phi(h, k) = 0$, the equation is satisfied only by $x = 0$, $y = 0$, i. e. it represents the point which is the origin; and if $\phi(h, k)$ be of the same sign as A and B, the equation can be satisfied by no real values of x and y.

Secondly, let m be positive, then A and B have contrary signs; and whatever be the sign of $\phi(h, k)$ i.e. whether it equals $+C$ or $-C$, the equation will be of one of the forms

$$\frac{Ay^2}{C} - \frac{Bx^2}{C} = 1, \text{ or } \frac{Bx^2}{C} - \frac{Ay^2}{C} = 1,$$

which represents a hyperbola with semi-axes $\sqrt{\frac{C}{A}}$, $\sqrt{\frac{C}{B}}$.

If $C = 0$, the equation is $y = \pm\sqrt{\frac{B}{A}}\,.\,x$, which represents two straight lines through the origin.

227. Next in the equation,

$$ay^2 + bxy + cx^2 + dy + ex + f = 0;$$

let the coefficients be such that $b^2 - 4ac = 0$, and that the numerators in the values of h and k are finite, then the coordinates of the center are infinite, which signifies that the curve has no center. In this case, as we cannot by changing the origin take away the terms in x and y, our first object must be to destroy the term involving the rectangle xy. For that purpose put

$$x'\cos\theta - y'\sin\theta \text{ for } x, \text{ and } x'\sin\theta + y'\cos\theta \text{ for } y,$$

and the equation becomes

$$Ay'^2 + Bx'^2 + (d\cos\theta - e\sin\theta)\,y' + (d\sin\theta + e\cos\theta)\,x' + f = 0,$$

the term involving $x'y'$ disappearing, as before, by the condition

$$\tan 2\theta = \frac{-b}{a-c},$$

which gives, since $b^2 = 4ac$,

$$\sin 2\theta = \frac{-b}{\sqrt{(a-c)^2 + b^2}} = \frac{-b}{a+c}$$

$$\cos 2\theta = \frac{a-c}{a+c},$$

taking the radical with the positive sign. Hence by means of the formulæ

$$\cos\theta = \sqrt{\tfrac{1}{2}(1+\cos 2\theta)},\ \sin\theta = \sqrt{\tfrac{1}{2}(1-\cos 2\theta)},\ \text{we get}$$

$$d\cos\theta - e\sin\theta = \frac{d\sqrt{a} - e\sqrt{c}}{\sqrt{a+c}} = D,$$

$$d\sin\theta + e\cos\theta = \frac{d\sqrt{c} + e\sqrt{a}}{\sqrt{a+c}} = E.$$

$$\text{Also } A = \tfrac{1}{2}\left\{a + c + \sqrt{(a-c)^2 + b^2}\right\} = a + c,$$

$$B = \tfrac{1}{2}\left\{a + c - \sqrt{(a-c)^2 + b^2}\right\} = 0,$$

therefore the equation becomes, suppressing the accents,

$$Ay^2 + Dy + Ex + f = 0,$$

$$\text{or } \left(y + \tfrac{1}{2}\frac{D}{A}\right)^2 = \frac{E}{A}\left(-x + \frac{D^2 - 4Af}{4AE}\right),$$

which represents a parabola, latus rectum $= \dfrac{E}{A}$, and co-ordinates of its vertex $x = \dfrac{D^2 - 4Af}{4AE}$, $y = -\tfrac{1}{2}\dfrac{D}{A}$, and axis parallel to the new axis of x. In this case the co-ordinates of the new origin cannot become infinite; for $A = a + c$ cannot become zero since a and c have the same sign; and if $E = 0$, then the transformed equation will no longer contain x; and being solved with respect to y, it will furnish two constant values for y, so that it will represent two parallel lines.

228. If in Art. 223 the coefficients of the proposed equation are such that one of the numerators $2ae - bd$ is zero, at the same time that $b^2 = 4ac$, (which two suppositions make the other numerator $2cd - be$ also vanish) both the co-ordinates of the center become indeterminate; the two equations (1) in that case are equivalent to a single independent equation, and the two lines which they represent, regarding h and k as the co-ordinates, coincide, and there

exists an infinite number of centers all situated in that line. The proposed equation, with the above relations among its coefficients, no longer in fact represents a curve, but two parallel straight lines; for, solving it, we get

$$y = -\frac{bx+d}{2a} \pm \frac{1}{2a}\sqrt{(b^2-4ac)x^2+2(bd-2ac)x+d^2-4af},$$

and this in the supposed case becomes

$$y = -\frac{bx+d}{2a} \pm \frac{1}{2a}\sqrt{d^2-4af},$$

and therefore represents two parallel straight lines, which are replaced by a single one if $d^2 = 4af$; and become altogether imaginary if $d^2 < 4af$.

General properties of Algebraical Curves.

229. The general equation of the n^{th} degree between x and y ought to contain all the combinations of the powers of x and y in which the sum of the indices does not exceed n; therefore when complete and arranged according to descending powers of y, it will be

$$a_0 y^n + (b_0 + b_1 x)\,y^{n-1} + (c_0 + c_1 x + c_2 x^2)\,y^{n-2} + \&c.$$
$$+ (l_0 + l_1 x + l_2 x^2 + \&c. + l_n x^n) = 0.$$

All equations between two variables x and y which can be reduced to this form, are called algebraical, all others are called transcendental; hence arises the distinction of lines into algebraical and transcendental, according as their equations are algebraical or transcendental.

230. The classification of lines in different orders according to the degrees of their equations would be to little purpose, if by changing the axes of the co-ordinates we altered the degree of the equation. But this is not the case. For having given, between x and y, the equation to a line referred to certain axes, in order to get the equation to the same line referred to new axes, we must replace x and y in the given equation by the values found in Art. 41; and as these values are of the first degree in x' and y', it

follows that the degree of the equation cannot be raised by this substitution. Neither can the degree of the transformed equation be less than that of the primitive equation; for if it could, then, by what has been proved, we could not return from it to the primitive equation, which is absurd.

231. The general equation of any degree comprehends not only all lines of the order expressed by that degree, but also all lines of inferior orders. Thus the above general equation of the n^{th} degree, by making $a_0 = b_1 = c_2 = \ldots = l_n = 0$, degenerates into the equation of the $(n-1)^{\text{th}}$. Also the equation of the second degree

$$(y - mx - c)(y - m'x - c') = 0$$

is clearly verified either by putting $y = mx + c$, or $y = m'x + c'$, which represent two lines of the first order; so that the proposed equation does not in reality represent a line of the second order at all, but two straight lines; or only one even of these, if $m = m'$, $c = c'$. Similarly, the equation of the third order

$$(y - mx - c)(ay - x^2) = 0$$

represents a line of the first order, and one of the second whose equation is $ay - x^2 = 0$. And in general according as a proposed equation of any degree is not, or is capable of being resolved into factors which are rational with respect to the variables x and y, it will represent a single line of the corresponding order, or several distinct lines of inferior orders.

232. A straight line cannot meet a curve of the n^{th} order in more than n points.

Let the co-ordinates be transformed so that the proposed line may be the axis of x, and let $V = 0$ be the resulting equation to the curve; in order to determine the points in which it is intersected by the straight line, we must put $y = 0$ in the equation $V = 0$, and the corresponding values of x will be the abscissæ of the required points. But $V = 0$ being of the n^{th} degree, the equation for determining x will be at the most of the n^{th} degree; therefore

x cannot have more than n values, and there cannot be more than n points of intersection; but there may be fewer than n, for the equation for determining x may be of a degree inferior to n, and may have equal or imaginary roots.

233. The general equation of the n^{th} degree between two variables, when complete, contains $1 + 2 + 3 + \&c. + (n + 1)$, or $\frac{1}{2}(n + 1)(n + 2)$, arbitrary constants, in which, since we may divide the whole equation by one of them, there is one superfluous which might be suppressed; consequently the number of independent constants is

$$\tfrac{1}{2}(n + 1)(n + 2) - 1, \text{ or } \tfrac{1}{2}n(n + 3).$$

Hence a curve of the n^{th} order may be made to fulfil $\frac{1}{2}n(n + 3)$ conditions; as, for instance, to pass through $\frac{1}{2}n(n + 3)$ points; for giving to x and y their values at each of the given points, we get $\frac{1}{2}n(n + 3)$ different equations by means of which the values of the constants may be determined. Hence a curve of the second order may pass through five given points.

234. To determine the conic section which passes through five given points.

Take the axes of the co-ordinates so that each contains two of the given points; and let y_1, y_2, be the ordinates of the points situated in the axis of y; x_1, x_2, the abscissæ of the points situated in the axis of x; and x_3, y_3, the co-ordinates of the fifth given point. Then substituting successively the co-ordinates of each of these points in the place of x and y in the general equation (where every coefficient is divided by the constant term),

$$ay^2 + bxy + cx^2 + dy + ex + 1 = 0,$$

we get the five equations

$$ay_1^2 + dy_1 + 1 = 0, \qquad ay_2^2 + dy_2 + 1 = 0,$$
$$cx_1^2 + ex_1 + 1 = 0, \qquad cx_2^2 + ex_2 + 1 = 0,$$
$$ay_3^2 + bx_3y_3 + cx_3^2 + dy_3 + ex_3 + 1 = 0,$$

which give for the five unknown quantities the values

$$a = \frac{1}{y_1 y_2}, \quad d = -\frac{y_1 + y_2}{y_1 y_2}, \quad c = \frac{1}{x_1 x_2}, \quad e = -\frac{x_1 + x_2}{x_1 x_2},$$

$$b = -\frac{1}{x_3 y_3}\left\{\frac{y_3(y_3 - y_2 - y_1)}{y_1 y_2} + \frac{x_3(x_3 - x_2 - x_1)}{x_1 x_2} + 1\right\}.$$

Now provided no three of the given points be in a straight line, none of the quantities x_1, x_2, &c. is zero; therefore the above values of a, b, &c. are neither infinite, nor indeterminate, and none of them has more than one value; therefore through five points, provided no three be in a straight line, a conic section, and only one, can be made to pass.

235. To find the position of the center of any curve.

The center of a curve is a point C (fig. 24), such that any chord of the curve PP' drawn through it, is bisected in it. (It must be observed, however, that if PP' meet the curve in more points than two, it is sufficient that these points combined in a certain order should be two and two equally distant from C). If the curve be referred to any two axes originating in C, and PN, $P'N'$ be the ordinates parallel to Cy of the extremities of a chord, we see from the equal triangles PCN, $P'CN'$, that these ordinates are equal and of contrary signs; the same thing is true for the abscissæ of P and P'; as well as for the extremities of every other chord passing through C. If therefore $\phi(x, y) = 0$ be the equation to the curve, and if it be satisfied by $x = a$, $y = b$, it must also be satisfied by $x = -a$, $y = -b$; that is, it must be such as not to alter when the signs of the two variables are changed; and conversely, if it have this property, the origin is the center of the curve. When $\phi(x, y) = 0$ is algebraic, it cannot have the above property unless the dimension of every term be even in an equation of an even degree, and odd in an equation of an odd degree; for in the former case the equation is not at all altered by replacing x and y by $-x$ and $-y$; and in the latter (in which case the equation cannot have a constant term) the sign of every

term will be altered, and therefore the whole equation unaltered. Hence to find whether a proposed curve admits of a center, we must refer it to parallel axes through a new origin having co-ordinates h, k, by putting $x = x' + h$, $y = y' + k$, and equate to zero the coefficients of all the terms which are of a dimension different (as far as regards odd and even) from the degree of the equation; if these conditions can all be satisfied by real and finite values of h and k, the curve has a center, and h and k are its co-ordinates; in the contrary case the curve has no center. Of this process we have an example at Art. 223.

236. The locus of the middle points of a system of parallel chords of any curve is called its diametral curve. If the curve be of the n^{th} order, the points of intersection with its ordinates real or imaginary will be in number n; and their combinations on the same indefinite line will form $\frac{1}{2} n (n - 1)$ different chords, and as many middle points, and therefore the diametral curve, since it may be met by an indefinite line in $\frac{1}{2} n (n - 1)$ points, will have an equation of the degree $\frac{1}{2} n (n - 1)$. For curves of the second order, since $n = 2$, the diametral curves can only be straight lines.

237. To find the locus of the middle points of a system of parallel chords of any curve.

Let the chords be parallel to a line through the origin whose equation is $y = mx$, and let $\phi(x, y) = 0$, be the equation to the curve; also let x', y', be the co-ordinates of the middle point of any one of these chords, and take it for the origin without altering the direction of the axes, and therefore put $x' + x$ for x, and $y' + y$ for y; then the transformed equation to the curve is $\phi(x' + x, y' + y) = 0$, and the equation to the chord is $y = mx$. Hence the values of x, corresponding to the points of intersection of the curve and chord, result from the equation $\phi(x' + x, y' + mx) = 0$, and because the origin bisects the chord, the values of x must be equal and of contrary signs; from which consideration we may obtain a relation between x' and y', which is the equation to the required locus.

238. Thus if $\phi(x, y) = 0$ be the general equation of the second order,

$$ay^2 + bxy + cx^2 + dy + ex + f = 0,$$

putting $x = x' + x$, and $y = y' + mx$, we get

$$a(y' + mx)^2 + b(x + x')(y' + mx) + c(x + x')^2$$
$$+ d(y' + mx) + e(x + x') + f = 0,$$

and because the values of x are to be equal and of contrary signs, the term involving the first power of x must disappear;

$$\therefore\ 2amy' + b(x'm + y') + 2cx' + dm + e = 0,$$
$$\text{or } y'(2am + b) + x'(2c + bm) + dm + e = 0,$$

the equation to a straight line. Hence there will be an infinite number of diameters corresponding to the various values of m. If the diameter is to be perpendicular to its chords, we must have

$$1 + m\left(-\frac{2c + bm}{2am + b}\right) = 0, \text{ or } m^2 + \frac{2(c - a)}{b}m - 1 = 0,$$

which will necessarily give two real values of m; hence there are generally two diameters which bisect their ordinates perpendicularly.

239. Not only do we distribute algebraical curves, into orders according to the degree of their equations; but we also investigate the different families of lines which may be comprised amongst those of the same order; and even the different species of each family, if necessary. The individual lines of the same family, or species, we then classify according to certain characteristics easy to be recognized, which completely distinguish them from one another; and we lastly endeavour to determine the form and properties of each of them. This we have effected for equations of the first and second degrees; the former gives only straight lines as has been said; the latter gives three species of curves very distinct, viz. the parabola which has no center; and the ellipse and hyperbola, both of which have a center, but only the latter has asymptotes.

The enumeration of lines of the third order was first made by Newton, who found 72 species comprised in 14 divisions; Stirling added 4 species which had been omitted, and, lastly, Cramer added 2 more, making in all 78 species.

---

240. The following are examples of finding the equations to various loci, and of tracing curves by means of their equations.

(1) Having given the base and altitude of a triangle, to find the locus of the center of the inscribed circle.

$SC = CH = c =$ half given base (fig. 41), $PN = a$ the given altitude, O the center of the inscribed circle, $CM = x$, $MO = y$ its co-ordinates, $CN = x'$; then

$$\tan S = \frac{a}{c+x'}, \quad \tan\frac{S}{2} = \frac{y}{c+x}, \quad \tan H = \frac{a}{c-x'}, \quad \tan\frac{H}{2} = \frac{y}{c-x};$$

$$\therefore \quad \frac{a}{c+x'} = \frac{2y(c+x)}{(c+x)^2 - y^2}, \quad \frac{a}{c-x'} = \frac{2y(c-x)}{(c-x)^2 - y^2};$$

therefore inverting and adding in order to eliminate x', we get the required equation, which is of the third degree,

$$\frac{2}{a} = \frac{1}{y} - \frac{y}{c^2 - x^2}.$$

(2) Having given the base of a triangle, and the sum of the other two sides, to find the locus of the center of the inscribed circle.

SH the given base $= 2ae$, $SP + PH = 2a$, and $SO = r$, $HSO = \theta$, (fig. 41), polar co-ordinates of the describing point O;

then area of triangle $= \frac{1}{2}$ perimeter $\times$ radius of inscribed circle

$$= a(1+e)\, r \sin\theta,$$

also area of triangle $= SP \sin 2\theta \,.\, ae = \dfrac{a^2 e(1-e^2)\sin 2\theta}{1 - e\cos 2\theta}$;

$$\therefore \quad r = \frac{2ae(1-e)\cos\theta}{1 - e\cos 2\theta},$$

the equation to a conic section of which SH is the major axis.

(3) Two given circles are traced upon a plane, and a line is drawn touching one and cutting the other in two points, at which tangents are drawn to the latter circle; to find the locus of the intersection of the tangents.

$$OP = r,\ POO' = \theta,\ OO' = c,\ OQ = a,\ O'Q' = a' \text{ (fig. 69)},$$

$$c \cos\theta - a' = ON = \frac{OQ^2}{OP} = \frac{a^2}{r};$$

$$\therefore\ r = \frac{a^2}{c\cos\theta - a'},$$

hence the locus is a conic section of which the centers of the circles are the foci.

(4) Two focal distances of a conic section include a constant angle, and one of them is produced to meet the tangent at the extremity of the other, to find the locus of the point of intersection.

$$AST = \theta,\ ST = r,\ PSQ = \alpha,\ SPT = \phi \text{ (fig. 30)},$$

$$\frac{SP}{ST} = \frac{\sin(\alpha + \phi)}{\sin\phi} = \cos\alpha + \sin\alpha \cot\phi$$

$$= \cos\alpha + \sin\alpha \frac{e\sin(\theta+\alpha)}{1 + e\cos(\theta+\alpha)} = \frac{\cos\alpha + e\cos\theta}{1 + e\cos(\theta+\alpha)},$$

$$\because\ \cot\phi = \tan SPG = \frac{GL}{PL} = \frac{e\,SP\sin PSA}{a(1-e^2)} \text{ (Art. 135)},$$

$$\text{and } SP = \frac{a(1-e^2)}{1 + e\cos(\theta+\alpha)};$$

$$\therefore\ r = \frac{a(1-e^2)}{\cos\alpha + e\cos\theta},$$

the equation to a conic section with focus S, and which is an ellipse, hyperbola, or parabola, according as $\cos\alpha >, <$ or $= e$.

(5) To find the equation to the curve traced out by a point in the perimeter of a circle which rolls upon another equal circle.

Let A' be the describing point, at first in contact with A, and AA' the curve traced out, (fig. 70); C, C' the centers of the circles; join AA', CC', and let

$$AA' = r,\ A'AE = \theta,\ AC = a.$$

AA' is manifestly parallel to CC', draw DA' parallel to AC, and therefore $= AC$;

$$\therefore\ AA' = CD = CC' - DC',$$

$$\text{or}\ \ r = 2a - 2a\cos\theta,\ \text{the polar equation.}$$

Or if $AR = x$, $RA' = y$, be the rectangular co-ordinates of A',

$$\sqrt{x^2 + y^2} = 2a\left(1 - \frac{x}{\sqrt{x^2 + y^2}}\right);$$

$$\therefore\ x^2 + y^2 = 2a\left(\sqrt{x^2 + y^2} - x\right).$$

(6) To find the equation to the curve described by a point in the perimeter of a circle which rolls within another circle of four times its radius.

P the describing point, at first in contact with A (fig. 70), and AP the curve traced out;

$$CN = x,\ PN = y,\ CA = 4a,\ QO = a,$$

O being the center of the rolling circle,

$ACD = \theta$; therefore $QOP = 4\theta$, and consequently, OM being perpendicular to AC,

$$\angle POM = \pi - \left(\frac{\pi}{2} - \theta\right) - 4\theta = \frac{\pi}{2} - 3\theta;$$

$$\therefore\ x = CM + Pn = 3a\cos\theta + a\cos 3\theta = 4a(\cos\theta)^3,$$

$$y = OM - On = 3a\sin\theta - a\sin 3\theta = 4a(\sin\theta)^3;$$

$$\therefore\ \left(\frac{x}{4a}\right)^{\frac{2}{3}} + \left(\frac{y}{4a}\right)^{\frac{2}{3}} = 1.$$

(7) To find the locus of the intersection of a tangent to a given curve and the perpendicular let fall upon it from a given point.

PY (fig. 71) a tangent to the given curve at a point whose co-ordinates are $SN = x$, $PN = y$, S being the given point; SY a perpendicular upon PY, $SM = x'$, $MY = y'$, the co-ordinates of Y. Then drawing YQ parallel to SN, from the similar triangles SMY, PYQ, $\dfrac{SM}{MY} = \dfrac{PQ}{QY}$

$$\text{or } \frac{x'}{y'} = \frac{y - y'}{x + x'} \text{ or } x'^2 + y'^2 = yy' - xx'.$$

Also if α be the angle which PY forms with the axis of x, $\tan\alpha = f(x, y)$ is known from the nature of the curve, but $\tan\alpha = \dfrac{x'}{y'}$;

$$\therefore \frac{x'}{y'} = f(x, y).$$

These two equations together with the equation to the curve will enable us to eliminate x and y, and the result will be the equation to the locus of Y.

Ex. 1. To find the locus of the intersection of the tangent to a parabola and the perpendicular upon it from the vertex.

$$y^2 = 4ax,$$

$$\tan\alpha = \frac{2a}{y} = \sqrt{\frac{a}{x}}; \quad \therefore \frac{x'}{y'} = \sqrt{\frac{a}{x}};$$

$$\therefore x = a\frac{y'^2}{x'^2} \text{ and } y^2 = 4a^2 . \frac{y'^2}{x'^2} \text{ or } y = 2a\frac{y'}{x'};$$

$$\therefore x'^2 + y'^2 = 2a\frac{y'^2}{x'} - a\frac{y'^2}{x'};$$

$$\text{or } y'^2 = \frac{x'^3}{a - x'}, \text{ the equation to the Cissoid of Diocles.}$$

But if the perpendicular be let fall from the focus, the equation to the parabola reckoning from that point is

$$y^2 = 4a(x+a); \quad \therefore \tan\alpha = \sqrt{\frac{a}{x+a}} = \frac{x'}{y'};$$

$$\therefore x = a\left(\frac{y'^2}{x'^2} - 1\right), \quad y = 2a\frac{y'}{x'};$$

$$\therefore x'^2 + y'^2 = \frac{a}{x'}(x'^2 + y'^2);$$

$$\therefore x' = a;$$

the equation to the line touching the parabola at its vertex.

Ex. 2. If the curve be an ellipse, and the perpendicular be let fall from the center, the equation to the locus of the foot of the perpendicular will be found to be

$$(x^2 + y^2)^2 = a^2x^2 + b^2y^2,$$

which agrees with the polar equation already found (Art. 136)

$$r^2 = a^2(1 - e^2\sin^2\theta).$$

In the case of the hyperbola, changing b^2 into $-b^2$, the equation is

$$(x^2 + y^2)^2 = a^2x^2 - b^2y^2,$$

which if $b = a$ becomes

$$(x^2 + y^2)^2 = a^2(x^2 - y^2),$$

representing a curve called the Lemniscata of Bernouilli, and whose polar equation is $r^2 = a^2\cos 2\theta$.

(8) If a curve roll upon an equal one, similar points being always in contact, to find the locus of any given point in the rolling curve.

Let AP be the fixed, and $A'P$ the rolling curve (fig. 71), S' the describing point, and S a point similarly situated in the fixed curve. Join SS' meeting the common tangent at P in Y; also join SP, $S'P$. Then because the points

in contact are similar, SP, $S'P$ are equal and equally inclined to the common tangent PY; therefore PY bisects SS' at right angles, and therefore the locus of S' is similar to that of Y, the foot of the perpendicular from S upon the tangent to the fixed curve.

If therefore $y = f(x)$ be the equation to the locus of Y, $\frac{y}{2} = f\left(\frac{x}{2}\right)$ is the equation to the locus of S'. Hence if the curves are equal parabolas, and S the focus, its locus will be a straight line; if S be the vertex, its locus will be the cissoid of Diocles; if the curves are ellipses, and S the focus, its locus will be a circle; if S be the center, the equation to its locus will be

$$x^2 + y^2 = 2\sqrt{a^2x^2 + b^2y^2}.$$

(9) To find the locus of the intersection of a normal to an ellipse and the perpendicular upon it from the center.

Let x', y' be the co-ordinates of P (fig. 46); then the equations to PF, CF are respectively

$$y - y' = \frac{a^2}{b^2}\frac{y'}{x'}(x - x'),$$

$$y = -\frac{b^2}{a^2}\frac{x'}{y'}\cdot x,$$

$$\text{also } \frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1;$$

and it remains to eliminate x' and y' between these equations. By means of the second the first is reduced to

$$y - y' = -\frac{x}{y}(x - x') \quad \text{or } x^2 + y^2 = xx' + yy' = xx'\left(1 - \frac{b^2}{a^2}\right),$$

$$\frac{x'}{a} = \frac{a}{x}\cdot\frac{x^2 + y^2}{a^2 - b^2};$$

$$\therefore \frac{y'}{b} = -\frac{b}{y}\cdot\frac{x^2 + y^2}{a^2 - b^2};$$

$$\therefore \left(\frac{a^2}{x^2}+\frac{b^2}{y^2}\right)\left(\frac{x^2+y^2}{a^2-b^2}\right)^2=\frac{x'^2}{a^2}+\frac{y'^2}{b^2}=1,$$

and the polar equation is $r=\dfrac{ae^2\sin\theta\cos\theta}{\sqrt{1-e^2\cos^2\theta}}$.

(10) To find the locus of the intersection of two tangents to an ellipse inclined to one another at a constant angle.

Let PW, $P'W$ be the tangents (fig. 44), CQ, CQ' perpendiculars upon them, $CW=r$, $TCW=\theta$, $QCQ'=2a$; also let $TCQ=\phi-a$, and consequently $TCQ'=\phi+a$, then (Art. 136) supposing $AC=1$,

$$r^2\cos^2(\phi+a-\theta)=1-e^2\sin^2(\phi+a),$$
$$r^2\cos^2(\phi-a-\theta)=1-e^2\sin^2(\phi-a);$$

subtracting these we get

$$r^2\sin 2(\phi-\theta)=e^2\sin 2\phi,\text{ or }\tan 2\phi=\frac{r^2\sin 2\theta}{r^2\cos 2\theta-e^2},$$

by means of which eliminating ϕ from either of the above equations, we find

$$\tan 2a\,(r^2+e^2-2)=2\sqrt{r^2(1-e^2\sin^2\theta)+e^2-1};$$

or in rectangular co-ordinates, putting $\tan 2a=m$,

$$m(x^2+y^2-a^2-b^2)=2\sqrt{a^2y^2+b^2x^2-a^2b^2}.$$

(11) To find the locus of the intersection of two normals to an ellipse at right angles to one another.

Let PF, $P'F'$ be the normals (fig. 43), CF, CF' perpendiculars upon them, $CO=r$, $BCO=\theta$, $BCF=\phi$, $BCF'=\dfrac{\pi}{2}-\phi$, $AC=1$; then

$$r\cos(\phi-\theta)=CF=\frac{e^2\cos\phi\sin\phi}{\sqrt{1-e^2\sin^2\phi}},$$

$$r\sin(\phi-\theta)=CF'=\frac{e^2\cos\phi\sin\phi}{\sqrt{1-e^2\cos^2\phi}},$$

adding the squares of these equations, we get

$$r^2 = \frac{e^4 (2 - e^2) \tan^2 2\phi}{(2 - e^2)^2 (1 + \tan^2 2\phi) - e^4},$$

or $\tan 2\phi = \dfrac{2n}{1 + n^2} \dfrac{r}{\sqrt{c^2 - r^2}}$, putting $n = \dfrac{1}{b}$, $c = \dfrac{1 - b^2}{\sqrt{1 + b^2}}$.

Again, dividing one equation by the other and reducing,

$$\cos 2(\phi - \theta) = \frac{1 - n^2}{1 + n^2} \cos 2\phi.$$

Hence substituting the above value of $\tan 2\phi$, we get

$$r = c \frac{\tan^2 \theta - n^2}{\tan^2 \theta + n^2}.$$

(12) To find the locus of the intersection of two normals to a parabola at right angles to one another.

If $\tan PGN = m$ (fig. 29), then $PN = 2am$ and $AN = am^2$; therefore the equation to PG is

$$y - 2am = -m(x - am^2),$$

and changing m into $-\dfrac{1}{m}$, the equation to a second normal perpendicular to PG is

$$y + \frac{2a}{m} = \frac{1}{m}\left(x - \frac{a}{m^2}\right).$$

Hence, adding and subtracting, we get

$$y = a\left(m - \frac{1}{m}\right),$$

$$x - 3a = a\left(m - \frac{1}{m}\right)^2;$$

$$\therefore\ y^2 = a(x - 3a),$$

the equation to the required locus, which is consequently a parabola.

(13) To find the locus of the intersection of two tangents to a parabola which include a given angle (fig. 29).

PY, py two tangents intersecting at an angle $YQy = YSy = a$, $SQ = r$, $ASQ = \theta$.

$$\text{Now } ASY = QYy = QSy = \theta - a - ASY;$$

$$\therefore\ ASY = \tfrac{1}{2}(\theta - a), \quad \text{and } QSY = \tfrac{1}{2}(\theta + a);$$

$$\therefore\ r\cos\tfrac{1}{2}(\theta + a) = SY = a\sec\tfrac{1}{2}(\theta - a);$$

$$\therefore\ r = \frac{2a}{\cos a + \cos\theta},$$

the equation to a conic section with focus S.

(14) To find the locus of the point which is the intersection of three normals to an ellipse.

The equation to the normal at a point (x', y') of an ellipse is

$$y - y' = \frac{a^2}{b^2}\frac{y'}{x'}(x - x'),$$

$$\text{or } yx'\sqrt{1 - e^2} = (x - e^2x')\sqrt{a^2 - x'^2}.$$

Let h, k be co-ordinates of a given point through which the normal passes, then

$$k^2x'^2(1 - e^2) = (h - e^2x')^2(a^2 - x'^2)\ldots\ldots\ldots(1)$$

is the equation for determining x', the abscissa of the point in the ellipse; and as this equation is of the form

$$e^4x'^4 - \text{\&c.} - a^2h^2 = 0,$$

it has two or four possible roots; and consequently through the point (h, k) in general either four or two normals can be drawn. If two of the possible roots become equal (which can only happen in the case of four real roots), then three normals will pass through the point (h, k); in that case the derived equation

$$k^2 x' (1 - e^2) = - e^2 (h - e^2 x') (a^2 - x'^2) - x' (h - e^2 x')^2$$

has one of them. Dividing this by (1), we find

$$\frac{1}{x'} = - \frac{e^2}{h - e^2 x'} - \frac{x'}{a^2 - x'^2}, \quad \text{or } h a^2 = e^2 x'^3;$$

$\therefore\ x' = \left(\dfrac{h a^2}{e^2}\right)^{\frac{1}{3}}$ satisfies equation (1), and substituting we get

$$k^{\frac{2}{3}} (1 - e^2)^{\frac{1}{3}} + h^{\frac{2}{3}} = (a e^2)^{\frac{2}{3}}$$

for the equation of condition that (1) may have equal roots, and as often as h and k satisfy this equation, three normals to the ellipse will pass through the point (h, k). The above is consequently the equation to the locus of the intersection of three normals to an ellipse; and coincides, as might have been foreseen, with the equation to the evolute.

On tracing curves from their equations.

241. When a curve passes through the origin, the angle at which it cuts the axis of x may be determined by taking the limit of $\frac{y}{x}$ when $x = 0$, which will be the value of the tangent of that angle.

Let AP be a curve passing through the origin A (fig. 72), P a point in it near A with co-ordinates $AN = x$, $NP = y$; draw the secant AP, then $\tan PAN = \frac{y}{x}$; now let P move up to and coincide with A, then the secant AP coincides with AT the tangent to the curve at A, and

$$\tan TAN = \text{limit of } \tan PAN = \text{limit of } \frac{y}{x}, \text{ when } x = 0.$$

Hence the angles at which a proposed curve cuts the co-ordinate axes may always be determined; for we have only to transfer the origin to one of the points in question, and in the transformed equation take the limit of $\frac{y}{x}$ by putting $x = 0$.

242. In tracing curves from their equations, whenever y is given or can be found in an explicit function of x, it will be best to use algebraical processes alone.

First, determine the points where the curve cuts the axis of x; transfer the origin of co-ordinates, if necessary, to one of these points, expand y in a series ascending by powers of x, and let

$$y = ax^m + bx^n + \&c.$$

If $m < 1$, limit of $\dfrac{y}{x} = \infty$; and therefore the curve is perpendicular to the axis of x, and is concave.

If $m = 1$, limit of $\dfrac{y}{x} = a$; therefore the curve cuts the axis at an angle $= \tan^{-1}(a)$, and is situated above or below the tangent, i. e. is convex or concave, according as b is positive or negative.

If $m > 1$, limit of $\dfrac{y}{x} = 0$, and the curve touches the axis of x, and is convex.

Similarly the form of the curve at all its other intersections with both the axes may be found.

Secondly, determine the nature of the infinite branches, and to that end expand y in a descending series of powers of x (on the supposition that both x and y are very great), and let

$$y = ax^m + bx^n + \dots\dots + ex + f + \frac{g}{x} + \dots\dots$$

$\therefore$ $y = ax^m + bx^n + \dots\dots + ex + f$ is the equation to the asymptotic curve, above or below which the given curve is situated, according as g is positive or negative.

If $m = 1$, the equation to the asymptote is $y = ex + f$, a straight line; and the curve will be situated above or below the asymptote, i. e. will be convex or concave to the axis of x, according as g is positive or negative.

In this case the infinite branch represented by

$$y = ex + f + \frac{g}{x} + \ldots\ldots$$

is said to be hyperbolic.

If $m > 1$, the infinite branch is parabolic, and is concave or convex, according as its asymptote is concave or convex.

Having thus found the figure of the curve at the points where it cuts the axes, and also when x and y are very great, it is easy to trace the intermediate parts. For the actual position of the maximum or minimum ordinates and points of contrary flexure, recourse must be had to the methods of the Differential Calculus.

If the equation to a curve can be resolved, and give for y the values V, V', &c. functions of x and constants, we must trace separately each of the curves represented by $y = V$, $y = V'$, &c., all of which will be particular branches of the proposed curve. The branches cannot terminate abruptly; they will either go on to infinity, or the ordinates will become imaginary, in which case two branches will be united and mutually continue one another.

Ex. 1. To trace the curve whose equation is $ay^3 = (a^2 - x^2)^2$ (fig. 73).

Since the equation does not alter when $-x$ is written for x, the curve is symmetrical with respect to the axis of y; also for any value of x, y has only one possible value, and is always positive. When $x = 0$, $y = a$, and as x increases y diminishes; therefore a is a maximum value of y, and the curve cuts the axis of y at B at right angles, and is concave to the axis of x. When $x = a$, $y = 0$, and the curve cuts the axis of x at right angles at A, because if we remove the origin to that point by making $x = a + x'$, we get $ay^3 = (2ax' + x'^2)^2$, and therefore limit of $\frac{y^3}{x'^3} =$ limit of $\frac{(2a + x')^2}{ax'} = \infty$. When $x > a$, the

equation becomes $ay^3 = (x^2 - a^2)^2$, and as x increases, y increases, till x is very large, when the relation between them approximates to

$$a^{\frac{1}{3}}y = x^{\frac{4}{3}}\left(1 - \frac{a^2}{x^2}\right)^{\frac{2}{3}} = x^{\frac{4}{3}}\left(1 - \frac{2}{3}\frac{a^2}{x^2}\right) = x^{\frac{4}{3}} - \frac{2}{3}\frac{a^2}{x^{\frac{2}{3}}};$$

$\therefore$ $a^{\frac{1}{3}}y = x^{\frac{4}{3}}$ is the equation to the asymptotic parabola ZOZ', below which the curve lies, because the second term of the expansion of y is negative. Hence the figure of the curve is that annexed, having a point of inflexion at P; for the curve is concave to the axis of x at A, and afterwards convex because the parabola with which it tends to coincide is so. There is of course another point of inflexion at P'.

Ex. 2. To trace the curve whose equation is

$$y = \frac{x - 3}{(x - 1)(x - 2)} \quad \text{(fig. 74)}.$$

When $x = 0$, $y = -\frac{3}{2}$, and the curve cuts the axis of y at D; as long as $x < 1$, y is negative and becomes infinite when $x = 1$; therefore the ordinate BE corresponding to $x = 1$ is an asymptote, and we thus get the branch DE.

When x is between 1 and 2, y is positive, and becomes very great both when x is a little less than 2 and a little greater than 1, and gives the portion $E'GF'$.

When $x = 2$, y is infinite, so that the ordinate FF' is an asymptote.

When x lies between 2 and 3, y is negative and gives the portion FH.

When $x = 3$, $y = 0$, and the curve cuts the axis at H.

When $x > 3$, y is positive and gives the portion HK; and when x is very great,

$$y = \frac{x}{x^2 - 3x} = \frac{1}{x - 3} = 0;$$

therefore the axis of x is an asymptote.

When x is negative, the equation is

$$y = -\frac{x+3}{(x+1)(x+2)};$$

therefore y is always negative, and diminishes as x increases, and becomes 0 when $x = \infty$ and gives the branch DL.

Ex. 3. To trace the curve whose equation is

$$ay = x^2 + x\sqrt{2ax - x^2} \quad \text{(fig. 75).}$$

Taking the radical, which of course admits of a double sign, first with a positive sign, we have when $x = 0$, $y = 0$, and limit of $\frac{y}{x} = 0$; therefore the curve passes through the origin A and touches the axis of x; when $x = a$, $y = 2a$, and when $x = 2a$, $y = 4a$, beyond which y is impossible; we thus get the portion of the curve AED, the ordinate CD being a tangent at D. Again taking the radical with a negative sign, so that

$$ay = x^2 - x\sqrt{2ax - x^2},$$

$x = 0$ gives $y = 0$, and limit of $\frac{y}{x} = 0$, as before; as long as $x < a$, y is negative; when $x = a$, $y = 0$, and putting $x' + a$ for x we get, supposing x' very small,

$$ay = ax' + \frac{3}{2}x'^2; \qquad \therefore \text{ limit of } \frac{y}{x'} = 1;$$

hence the portion of the curve AFB touches the axis of x at A, and cuts it at an angle of 45^0 at B, and is there situated above the tangent. When $x = 2a$, $y = 4a = DC$, and for greater values of x, y is impossible; hence DC is a tangent to both portions of the curve at D, the point in which they are united. For all negative values of x, y is impossible. The curve is therefore such as is represented, the tangent being parallel to the axis of x at F and E.

THE END.

ERRATA.

Page 13	line 6	*for*	equation	*read*	equations,
.... 26	 16	...	of x	......	of P,
.... 36	 21	...	$P'N$	......	$P_1 N$,
.... 61	 20	...	by b^2	......	by a^2,
.... 62	 4	...	up and	......	up to and.

STUDENTS reading this work for the first time may confine their attention to the following Articles:

1–28, 45–48, 55–57, 68–73, 75–82, 104–111, 115, 117–127, 155–162.

THE

FIRST THREE SECTIONS

OF

NEWTON'S PRINCIPIA,

WITH

AN APPENDIX; P49

AND THE

NINTH AND ELEVENTH SECTIONS.

EDITED BY

JOHN H. EVANS, M.A.

LATE FELLOW OF ST JOHN'S COLLEGE, CAMBRIDGE,
AND HEAD MASTER OF SEDBERGH GRAMMAR SCHOOL, YORKSHIRE.

THIRD EDITION.

CAMBRIDGE:
PRINTED AT THE UNIVERSITY PRESS,
FOR J. AND J. J. DEIGHTON, CAMBRIDGE;
AND RIVINGTONS; AND LONGMAN AND CO., LONDON.

M.DCCC.XLIII.

The following pages with a few alterations were originally taken from the Manuscripts, which had been used in St John's College, and were printed with the view of saving to the Student the trouble of copying them. The few Propositions of the Seventh and Eighth Sections, now generally read in the University, are given in the Appendix, which also contains some examples of the applications of the principles of the first three Sections.

SEDBERGH,
January, 1843.

SECTION I.

OF THE METHOD OF LIMITS AND LIMITING RATIOS.

DEF. 1. THE *Limit* of a continually increasing or decreasing quantity or ratio is that quantity or ratio, to which it continually approximates, but to which, though it may approach nearer than by any assignable difference, it never becomes actually equal.

OBS. The limit of a varying quantity or ratio is frequently called the *ultimate* value of that quantity or ratio; when we say that one quantity is *ultimately* equal to another, it is not to be inferred that the two quantities are ever equal, though their difference may be less than any assignable quantity.

DEF. 2. Quantities or the ratios of quantities tend continually to equality, when the ratio of the difference to either of them continually decreases.

LEMMA I.

Quantities and the ratios of quantities, which tend continually to equality, and whose difference may be made to bear to either of them a ratio less than any finite ratio, have their limits equal.

For if the limits be not equal, let L and $L+D$ represent them; then the ratio of the difference of the limits to one of them

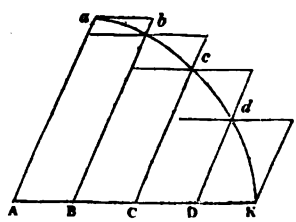
$$= D : L \quad \text{or} \quad D : L + D.$$

Now since the quantities or ratios tend continually to equality, the ratio of their difference to either of them must always be greater than that of the difference of their limits to either of the limits, that is than $D : L$ or $D : L + D$, either of which is a finite ratio. But by the hypothesis the ratio of their difference to either of them may be made less than any finite ratio, which is absurd; therefore the limits are not unequal, that is, they are equal.

Cor. Hence if the quantities or ratios be finite, the limit of their difference, as they tend continually to equality, must equal 0. If they be indefinitely great, the limit of their difference may be a finite quantity or ratio, for it would bear to either of them an indefinitely small ratio. Lastly, if they be indefinitely small, it must be a quantity or ratio, which vanishes compared with either of them.

Lemma II.

If in any figure AKa, bounded by the straight lines Aa, AK, and the curve line Ka, there be inscribed any number of parallelograms Ab, Bc, Cd....on equal bases AB, BC, CD,..., and the parallelograms Ba, Cb, Dc....be completed; then if the number of these parallelograms be increased and their breadths diminished indefinitely, the limit of the sum of each series will be the curvilinear area AKa.



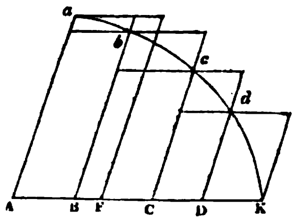
For as their bases are diminished, each series of parallelograms continually approximates to the area AKa. Also the difference between the two series is the sum of the parallelograms ab, bc, cd... which sum is equal to the parallelogram aB, for the base of each is equal to AB, and the sum of their altitudes to that of aB, and by diminishing the bases this difference, and therefore, *a fortiori*,

$X \to D$
$Y \to L$

the difference of either series and the area AKa may be made less than any assignable quantity, and therefore, by Lemma 1, the limit of either series is the curvilinear area AKa.

Lemma III.

If the two series of parallelograms be described in the same manner as in the last Lemma, except that their bases are not all equal, the limit of each series, when their bases are diminished indefinitely, is in this case also the curvilinear area AKa.



For take AF equal to the greatest base, and complete the parallelogram Fa; then this parallelogram, which is evidently greater than the difference between the two series of parallelograms, may, by diminishing the base, be made less than any assignable quantity. Hence the difference between the two series, and therefore *a fortiori*, the difference between each series and the area AKa may be made less than any assignable quantity; and they tend continually to equality, therefore, by Lemma 1, the limit of each series is the curvilinear area AKa.

Cor. 1. If the chords ab, bc, cd... be drawn, the limit of the area bounded by Aa, AK and the chords, when the bases AB, BC, CD.... are diminished indefinitely, is the curvilinear area AKa, for it always lies between this area, and the inner series of parallelograms.

Cor. 2. The limit of the figure bounded by Aa, AK and the tangents through a, b, c... is the same curvilinear area, since it lies always between the curvilinear area, and the outer series of parallelograms.

Cor. 3. The curve line aK is the limit of the boundary formed by the chords.

LEMMA IV.

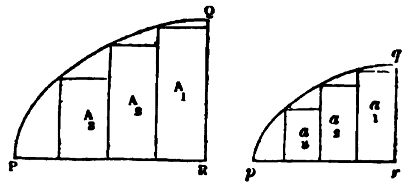
If in two curvilinear figures there can be inscribed the same number of parallelograms, which, when their number is increased, and their breadths diminished indefinitely, are ultimately to each other in a given ratio, the areas of the curvilinear figures will be in that ratio.

A

Let PQR, pqr be the figures, and let the parallelograms $A_1, A_2, A_3, \ldots$ be inscribed in the one, and $a_1, a_2, a_3, \ldots$ in the other,

and let $\dfrac{A_1}{a_1} = m + x_1$, $\dfrac{A_2}{a_2} = m + x_2$, $\dfrac{A_3}{a_3} = m + x_3$, &c. = &c.

$x_1 x_2 x_3 \ldots$ being quantities, which vanish, when the breadths of the parallelograms are diminished indefinitely, so that according to the hypothesis,



$$\text{limit } \frac{A_1}{a_1} = m = \text{limit } \frac{A_2}{a_2} = \text{limit } \frac{A_3}{a_3} = \&c.$$

Hence $A_1 = m a_1 + x_1 a_1$, $A_2 = m a_2 + x_2 a_2$, $A_3 = m a_3 + x_3 a_3$, &c. = &c.

$$\therefore\ A_1 + A_2 + A_3 + \ldots = m(a_1 + a_2 + a_3 + \ldots) + x_1 a_1 + x_2 a_2 + x_3 a_3 + \ldots$$

$$\therefore\ \frac{A_1 + A_2 + A_3 + \ldots}{a_1 + a_2 + a_3 + \ldots} = m + \frac{x_1 a_1 + x_2 a_2 + x_3 a_3 + \ldots}{a_1 + a_2 + a_3 + \ldots}$$

$$\therefore\ \text{limit } \frac{A_1 + A_2 + A_3 + \ldots}{a_1 + a_2 + a_3 + \ldots} = m + \text{limit } \frac{x_1 a_1 + x_2 a_2 + x_3 a_3 + \ldots}{a_1 + a_2 + a_3 + \ldots}$$

Now since x_1 vanishes in the limit, $x_1 a_1$ vanishes compared with a_1; similarly $x_2 a_2$ vanishes compared with a_2, $x_3 a_3$ compared with a_3, and so on; the number of terms also in the two series is the same, therefore ultimately $x_1 a_1 + x_2 a_2 + x_3 a_3 + \ldots$ vanishes compared with $a_1 + a_2 + a_3 + \ldots$,

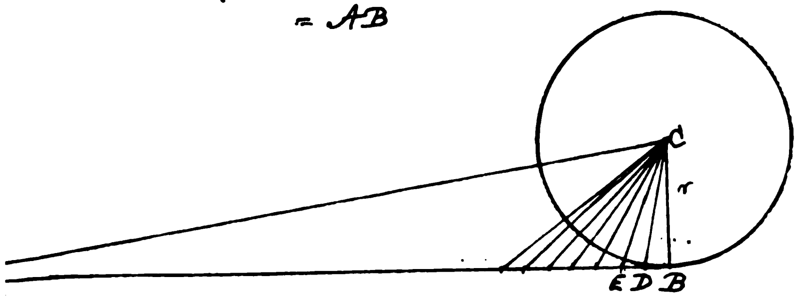
$$\text{or limit } \frac{x_1 a_1 + x_2 a_2 + x_3 a_3 + \ldots}{a_1 + a_2 + a_3 + \ldots} = 0.$$

To find the area of a circle.

r = radius

$\odot^{ce}$ of $\odot^{le} = 2\pi r$

$= AB$



$DB = ED$, &c.

When DB is indefinitely small,

Area of $\triangle ABC$ = area of $\odot^{le}$

$= \frac{1}{2}$ base $\times \perp^r$ height

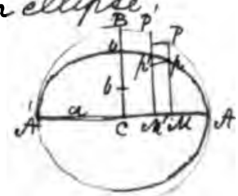
$= \frac{1}{2}(2\pi . r) . r$

$= \pi . r^2$

To find the area of an ellipse.

Let $\frac{pM}{PM} = \frac{bC}{BC}$ or $\frac{b}{a}$

then the curve $A'bpA$ is an ellipse.



$\frac{\text{Area of } \square\, p'M'}{\text{Area of } \square\, PM'} = \frac{pM}{PM}$; and, by Newton's Fourth Lemma the elliptic area is to the circular area in the same proportion.

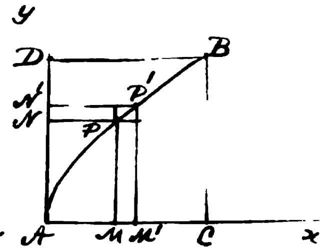
$\therefore \frac{\text{Area of } \theta^{e}}{\text{Area of } \odot^{le}} = \frac{b}{a}$

$\therefore$ Area of $\theta = \pi . a^2 \times \frac{b}{a}$

$= \pi ab$

To find the area of a parabolic sector.

$y^2 = 4mx$



Let $PM = y$, $P'M' = y'$
$AM = x$ $AM' = x'$

Area of $\square$ $PM' = MM' \times PM$
$= (x'-x)y$

But Area of $\square$ $PN' = PN \times NN'$
$= x(y'-y)$

$y'^2 = 4mx'$
$y^2 = 4mx$

$$y'^2 - y^2 = (y'-y)(y'+y) = 4m(x'-x)$$

$$y'-y = \frac{4m(x'-x)}{y'+y}$$

$$x(y'-y) = \frac{4mx(x'-x)}{y'+y} = \frac{y^2(x'-x)}{2y} \text{ ultimately}$$

$$= \frac{y(x'-x)}{2} \quad \text{"} \quad \text{"}$$

Or Area of $\square$ $PN' = \frac{1}{2}$ area of $\square$ PM'

Therefore, by Newton's Fourth Lemma,

Area of $ABD = \frac{1}{2}$ area of ABC

Or, parabolic area $ABC = \frac{2}{3}$ area of $\square$ $ADBC$.

$$\text{Also limit } \frac{A_1 + A_2 + A_3 + \ldots}{a_1 + a_2 + a_3 + \ldots} = \frac{\text{area } PQR}{\text{area } pqr},$$

$$\therefore \frac{\text{area } PQR}{\text{area } pqr} = m.$$

Cor. If there be two quantities of any kind, which are divided into the same number of parts, and if these parts, when their number is continually increased and the magnitude of each continually diminished, be to each other in a given ratio, the whole quantities will be in that ratio.

For if the parts be substituted for the parallelograms, and the whole quantities for the figures PQR, pqr, the reasoning will be the same in the two cases.

Def. 1. A *curve* is a line traced out by a moving point, which is continually changing the direction of its motion.

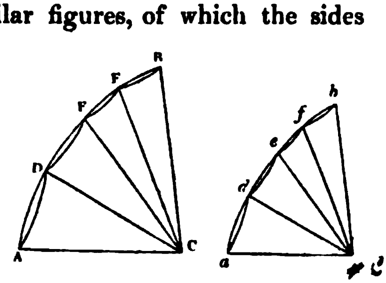
Def. 2. One curvilinear figure is said to be similar to another, when any rectilinear figure being inscribed in the first, a similar rectilinear figure may be inscribed in the other.

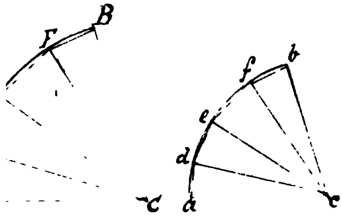
Obs. The curves and curvilinear figures, treated of in this Section, are always supposed to lie in one plane.

Lemma V.

The homologous sides of all similar curvilinear figures are proportionals, and their areas are in the duplicate ratio of the sides.

Let ACB, acb be two similar figures, of which the sides AB, AC, BC, are homologous to ab, ac, bc, respectively; then by definition, if $ADEBC$ be a polygon inscribed in ABC, a similar polygon $adebc$ may be inscribed in abc. Let such polygons be inscribed; and join CD, CE, &c. cd, ce, &c. dividing the polygons into the same number of similar triangles.





$$\therefore AD : AC = ad : ac,$$

$$\text{alt}^{do}\ AD : ad = AC : ac,$$

$$\text{Similarly } DE : de = DC : dc = AC : ac,$$

$$EF : ef = AC : ac,$$

....

therefore, componendo

$$AD + DE + EF + \&c. : ad + de + ef + \ldots = AC : ac.$$

Now this being always true, will be true when the number of sides is increased, and their magnitudes diminished, without limit;

$$\therefore \text{limit } AD + DE + EF + \ldots : \text{limit } ad + de + ef + \ldots = AC : ac,$$

and therefore by Lem. III, Cor. 3,

$$ADB : adb = AC : ac$$

$$= BC : bc.$$

Again, polygon $ADEBC$: polygon $adebc = AC^2 : ac^2$, and this being always true will be true in the limit;

$$\therefore \text{limit polygon } ADEBC : \text{limit } adebc = AC^2 : ac^2;$$

therefore by Lem. III, Cor. 1,

$$\text{curvilinear figure } ABC : \text{curvilinear fig. } abc = AC^2 : ac^2$$

$$= \overline{ADB}|^2 : \overline{adb}|^2$$

$$= BC^2 : bc^2.$$

Cor. If ACB, acb be two similar figures, and CE, ce be equally inclined to AC, ac, then $AC : CE = ac : ce$. Hence also this definition:

$$\frac{AD}{AC} = \frac{ad}{ac}, \quad \frac{DE}{AC} = \frac{de}{ac}, \quad \&c., \&c.$$

$$\frac{AD + DE + EF + \cdots}{AC} = \frac{ad + de + ef + \cdots}{ac}$$

Now this being always true is true when the number of sides is indefinitely increased

$$\therefore \quad \frac{\overparen{ADEFB}}{AC} = \frac{\overparen{adefb}}{ac}$$

and similarly, $\dfrac{\overparen{ADEFB}}{BC} = \dfrac{\overparen{adefb}}{bc}$

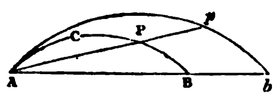
$$\frac{\text{Area of polygon } ADEBC}{\text{.......... } adebc} = \frac{\overline{AC}^2}{\overline{ac}^2} = \frac{\overline{BC}^2}{\overline{bc}^2}.$$

$$\frac{\text{Curv}^{l} \text{ area } ADBC}{\text{.......... } adbc} = \frac{\overline{AC}^2}{\overline{ac}^2} = \frac{\overline{BC}^2}{\overline{bc}^2} = \frac{\overparen{ADB}^2}{\overparen{adb}^2}$$

All circles are similar; all parabolas are similar.

Two curves are said to be similar, when there can be drawn in them two distances from two points similarly situated, such, that if any two other distances be drawn equally inclined to the former, the four are proportional.

Prob. Let the chord AB of the curve ACB be produced to b, to describe on Ab a curve similar to ACB.



In ACB take any point P, join AP, and produce AP to p, so that $Ap : Ab = AP : AB$; then if the curve Apb be the locus of all points, whose position is determined in the same manner as that of p, it will be similar to the curve APB.

Def. 1. The *tangent* to a curve AB at A is the straight line, in which the generating point would move, if instead of changing the direction of its motion it moved on in the direction which it had at A.

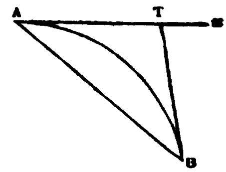
Def. 2. The curvature of a curve is said to be *continued* through a point, when the curve is wholly convex to the tangent at that point, and on the same side of it, and when the change of direction is not abrupt, but gradual; that is, if ATU, BT, (Fig. Lem. vi.) be tangents at A and B, in a curve of continued curvature, the angle BTU as B moves up to A, diminishes through every change of magnitude from its original value and ultimately vanishes.

Lemma VI.

If ACB be an arc of continued curvature, AB the chord, and ATU the tangent at A, the angle BAT between the chord and tangent, as B moves along the curve towards A and ultimately coincides with that point, continually diminishes and ultimately vanishes.

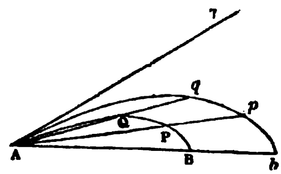
Let the tangents at A and B meet in the point T; then the angle BTU measures the change in the direction of the

motion of the generating point which takes place in passing from B to A, and since the curvature is continued, this angle, as B moves towards and ultimately coincides with A, continually diminishes and ultimately vanishes, therefore *a fortiori* the interior angle BAT continually diminishes and ultimately vanishes.



Cor. Similar conterminous arcs, which have their chords coincident, have a common tangent.

Let the similar conterminous arcs, APB, apb have their chords AB, Ab coincident, and let APp, AQq be any other coincident chords; then since the curves are similar $AP : Ap = AB : Ab = AQ : Aq$, therefore the arcs AP, Ap are similar, that is, the chords of the similar arcs AP, Ap coincide. Now let P and p move up to A, the arcs AP, Ap, since they are always similar, will vanish together, and APp in its ultimate position will be a tangent to each, that is, the arcs APB, Apb have a common tangent.

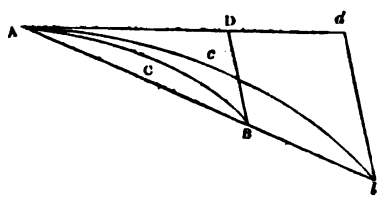


Def. The *subtense* of an arc is a straight line drawn from one extremity of the arc to meet at a finite angle the tangent to the arc at its other extremity.

Lemma VII.

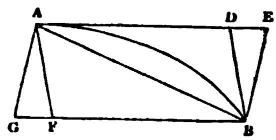
If BD be a subtense of the arc ACB of continued curvature, the chord AB, the arc ACB, and the tangent AD, when BD moves parallel to itself up to A, are ultimately equal to each other.

Produce AD to any fixed point d, and draw db parallel to DB to meet AB produced in b; on Ab describe the arc Acb similar to ACB, and as B moves up to A, let Acb so alter its form as to be always



similar to ACB; hence the two arcs have a common tangent, and the three lines AB, ACB, AD are always proportional to Ab, Acb, Ad. Now as B moves up to A, the angle bAd continually diminishes and ultimately vanishes, (Lemma VI.), the point b moves up to and coincides with d, and therefore Ab and Ad are ultimately equal; also, since db ultimately vanishes, $Ad + db$ is ultimately equal to Ab, therefore the arc Acb, which is greater than Ab and less than $Ad + db$, is ultimately equal to Ab. Hence AB, ACB, AD, which are always proportional to Ab, Acb, Ad, are ultimately equal to each other.

Cor. 1. Since the proof holds whatever be the inclination of BD to the tangent, provided it be finite, if BE be a subtense making any other finite angle with AD, the tangents AE, AD, and the chord and arc are ultimately equal.

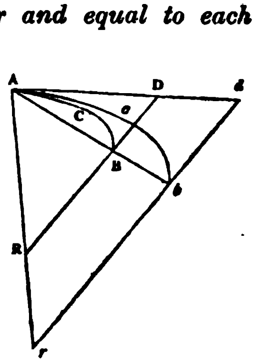


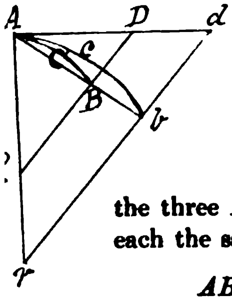
Cor. 2. Also if the parallelograms $ADBF$, $AEBG$ be completed, since AD, AE are always equal to BF, BG respectively, the lines AD, AE, BF, BG are ultimately equal to the chord and arc; and in all geometrical investigations the ultimate values of all these lines may be used indiscriminately for each other.

Lemma VIII.

If the straight lines AR, DBR, which meet in R, make with the chord AB, the arc ACB, and the tangent AD, the triangles ABR, $ACBR$, ADR; these three triangles, when B moves up to A, are ultimately similar and equal to each other.

Produce AD to a fixed point d, and draw dbr parallel to DBR, meeting AB, AR produced in b, r. On Ab describe the arc Acb similar to ACB, and let it so alter its form, as B moves up to A, as to be always similar to ACB. Then the two arcs will have a common tangent ADd, and the three triangles ABR, $ACBR$, ADR will be always similar to





the three Abr, $Acbr$, Adr respectively, and will bear each to each the same ratio, viz. that of $RA^2 : rA^2$; hence, alternando,

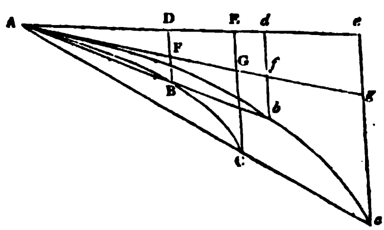
$$ABR : ACBR : ADR = Abr : Acbr : Adr.$$

Now let BD move parallel to itself up to A, then the angle bAd continually diminishes and ultimately vanishes; and Ab and therefore the intermediate arc Acb ultimately coincide with Ad; hence the triangles Abr, $Acbr$, are ultimately similar and equal to Adr; therefore the triangles ABR, $ACBR$, ADR, which are always proportional to them, are ultimately similar and equal to each other.

Obs. In the Lemma RBD is supposed to move parallel to itself towards A, that is, b moves along rd fixed, and the triangles Abr, $Acbr$, Adr are always finite; but the same thing will be true if RBD revólve round R fixed, in which case also, though r moves off to an infinite distance and the triangles Abr, $Acbr$, Adr increase indefinitely, they will be ultimately similar and equal to each other.

Lemma IX.

If the right line AE and the arc ABC, given in position, cut each other in a finite angle at A, and the ordinates BD, CE be drawn, making any other given angle with AE; when BD, CE move parallel to themselves up to A, the limiting ratio of area ABD : area ACE equals that of $AD^2 : AE^2$.



Produce AE to a fixed point e, and take Ad in Ae such, that $Ad : Ae = AD : AE$. Draw db, ec parallel to DB, or EC, meeting the chords AB, AC produced in b, c; and on Ac describe an arc similar to ABC: this arc shall pass through b, for by similar triangles and by construction,

$$AB : Ab = AD : Ad = AE : Ae = AC : Ac,$$

and therefore (Cor. Lemma v.) b is a point in the arc. As B and C move up to A, let the curve Abc so alter its form as to

Take e fixed, and $\dfrac{Ad}{Ae} = \dfrac{AD}{AE}$

By similar Δs, $\dfrac{Ab}{AB} = \dfrac{Ad}{AD} = \dfrac{Ae}{AE} = \dfrac{Ac}{AC}$

$\therefore$ (Cor. Lemma V) b is a point in the curve

$$\frac{\text{Area of } \Delta\, Abd}{\cdots\cdots\ \Delta\, ABD} = \frac{Ad^2}{AD^2} = \frac{Ae^2}{AE^2} = \frac{\text{Area of } \Delta\, Ace}{\cdots\cdots\ \Delta\, ACE}$$

$$\frac{\text{Area of } \Delta\, ABD}{\cdots\cdots\ \Delta\, ACE} = \frac{\text{Area of } \Delta\, Abd}{\cdots\cdots\ \Delta\, Ace}$$

$$\text{Limit } \frac{\Delta\, Abd}{\Delta\, Ace} = \frac{\Delta\, Afd}{\Delta\, Age} = \frac{Ad^2}{Ae^2}$$

$$= \text{Limit } \frac{\Delta\, ABD}{\Delta\, ACE} = \frac{AD^2}{AE^2}$$

Space described in time represented by AD is the curvilinear area ADd; in time AK, area AKk

$$\frac{\text{Space in time } AD}{\text{.......... } AK} = \frac{\text{area } ADd}{\text{area } AKk}$$

Since the force is finite it will only produce finite velocity in a finite time; therefore if we draw a tangent at A,

$\tan KAT = \frac{KT}{AK}$ is finite

By Lemma IX, $\mathcal{L}\, \frac{\text{area } ADd}{\text{.... } AKk} = \frac{AD^2}{AK^2}$

or the spaces vary as the squares of the times in the limit or the beginning of the motion.

be always similar to ABC, then the area ABD will be always similar to Abd, and ACE to Ace. Hence

$$\text{area } ABD : \text{area } Abd = AD^2 : Ad^2 = AE^2 : Ae^2$$
$$= \text{area } ACE : \text{area } Ace,$$
$$\therefore \text{area } ABD : \text{area } ACE = \text{area } Abd : \text{area } Ace.$$

Also the two arcs being similar have a common tangent at A, let this be $AFGfg$; and let BD, CE move parallel to themselves up to A; then the angle cAg continually diminishes and ultimately vanishes, and therefore

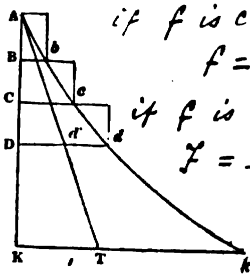
$$\text{L.R.* area } Abd : \text{area } Ace = \cancel{\text{L.R.}}\ \triangle\, Afd : \triangle\, Age$$
$$= \text{L.R. } Ad^2 : Ae^2.$$
$$\text{Hence L.R. area } ABD : \text{area } ACE = \text{L.R. area } Abd : \text{area } Ace$$
$$= \text{L.R. } Ad^2 : Ae^2$$
$$= \text{L.R. } AD^2 : AE^2.$$

Lemma X.

The spaces, described from rest by a body acted on by any finite force, are in the beginning of the motion as the squares of the times, in which they are described.

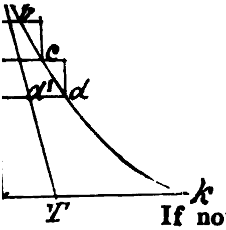
Def. A finite accelerating or retarding force is such, that the ratio of the time to the velocity generated or destroyed in that time is finite.

In Dynamics $S = \frac{1}{2}$
if f is consta
$f = \frac{2S}{t^2}$.
if f is varia
$f = L.R.$



Let the straight line AK represent the time of the body's motion from rest, and Kk, drawn at right angles to AK, the last acquired velocity; suppose the time divided into equal intervals AB, BC, CD &c., and let Bb, Cc, Dd &c., drawn at right angles to AK, represent the velocities acquired in the times AB, AC, AD &c.; let $Abcdk$ be the curve passing through the extremities of all the ordinates thus drawn; and complete the parallelograms Ab, Bc, Cd &c.

* L.R. signifies "limit of the ratio" or "limiting ratio."



If now the force be supposed to act by impulses, which would cause the body to move uniformly during the times AB, BC, CD &c., with the velocities Bb, Cc, Dd &c. respectively, the spaces described in the 1st, 2d, 3d &c. intervals will be represented by the parallelograms Ab, Bc, Cd &c. On this supposition therefore, the space described in time AD : space in time AK = sum of the parallelograms in the former case : sum in the latter; and this being true always, will be true when the intervals are diminished and their number increased indefinitely, in which case the force, which was supposed to act by impulses, approximates to a continued force, and the sums of the parallelograms to the areas ADd, AKk, as their limits.

Hence

space in time AD : space in time AK = area ADd : area AKk.

Let the tangent at A cut Kk in T; now, the force being finite, the ratio $AK : Kk$ is always finite; $\therefore AK : KT$, which equals L.R. $AK : Kk$ is a finite ratio, and therefore,

$$\tan KAT \left(= \frac{KT}{KA}\right) \text{ is finite,}$$

or KA makes a finite angle with the curve at A;

Hence by Lemma IX.

$$\text{L.R. area } ADd : \text{area } AKk = \text{L.R. } AD^2 : AK^2,$$

and therefore in the beginning of the motion, space $\propto$ (time)^2.

Cor. 1. Force is measured by the velocity generated in any time divided by the time, the force being supposed to remain constant for that time. Hence if Dd' be the velocity generated by the force at A, continued constant, in time AD,

$$F \text{ at } A = \frac{Dd'}{AD},$$

$s = \frac{1}{2} f t^2, \quad \propto t^2$

$f = \frac{2s}{t^2}$

force at $A = \frac{Dd'}{AD}$

$= \mathcal{L} \frac{Dd}{AD}$

$= \frac{KT}{AK}$

$= \frac{KT \cdot AK}{AK^2}$

$= \frac{2 \text{ area } \Delta AKT}{AK^2}$

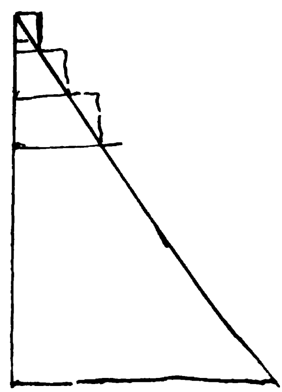
$= \mathcal{L} \frac{2 \text{ area } AKk}{AK^2}$

$= 2 \mathcal{L} \frac{\text{space}}{(\text{time})^2}$

If the force is constant, velocities are proportional to the times.

By Newton's x^{th} Lemma

Force uniform $= \frac{Dd}{AD}$



space $= \Delta ADd$

$= \frac{1}{2} AD \cdot Dd$

$= \frac{1}{2} \text{ time} \times bd$

$= \frac{1}{2} t \times ft$

$= \frac{1}{2} f t^2$

and this being always true, will be true when AD is diminished indefinitely,

$$\therefore\ F = \text{limit}\,\frac{Dd'}{AD} = \text{limit}\,\frac{Dd}{AD}$$

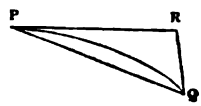
$$= \frac{KT}{AK} = \frac{KT.AK}{AK^2} = \frac{2\ \text{triangle}\ AKT}{AK^2} = 2\ \text{limit}\,\frac{\text{area}\ AKk}{AK^2}$$

$$= 2\ \text{limit}\,\frac{\text{space}}{(\text{time})^2}.$$

Cor. 2. The effect produced by F upon the body is independent of any motion which it may have, when F begins to act upon it. Hence generally if S be the space, through which a force F, acting on a body moving in any orbit, draws the body in T'' from the place it would have occupied if the extraneous force had not acted, $F = 2\ \text{limit}\,\frac{S}{T^2}$.

On the Curvature of Curve Lines.

Prop. I. If in PR, pr tangents at the points P, p in the curves PQ, pq, PR be taken equal to pr, and the subtenses QR, qr be drawn equally inclined to them, then when QR, qr move parallel to themselves to P, p,



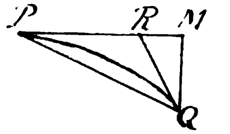
$$\frac{\text{curvature of}\ PQ\ \text{at}\ P}{\text{curvature of}\ pq\ \text{at}\ p} = \text{limit}\,\frac{QR}{qr}.$$

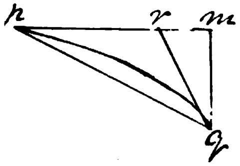
Draw the chords PQ, pq,

$$\text{then}\ \frac{\text{curvature of}\ PQ\ \text{at}\ P}{\text{curvature of}\ pq\ \text{at}\ p} = \frac{\text{angle of contact at}\ P}{\text{angle of contact at}\ p}$$

$$= \text{limit}\,\frac{\text{angle}\ QPR}{\text{angle}\ qpr}$$

draw perpendiculars QM, qm.





$$\frac{\text{curvature of } PQ \text{ at } P}{pq\text{-}p} = \text{limit}\,\frac{\sin QPR}{\sin qpr} = L.R\,\frac{\frac{QM}{PQ}}{\frac{qm}{pq}}$$

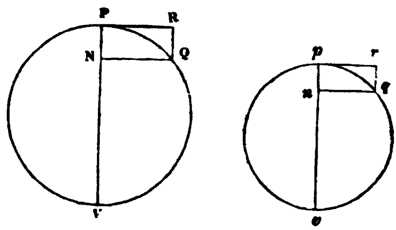
$$= \text{limit}\,\frac{\frac{QR}{QP}\sin R}{\frac{qr}{qp}\sin r} = L.R\,\frac{QM}{qm}\times\frac{pm}{PM} = L.R\,\frac{QR}{qr}\times\frac{pr}{PR}$$

and $PR = pr$.

$$= \text{limit}\,\frac{QR}{qr}. \qquad \therefore = L.R\,\frac{QR}{qr}$$

Prop. II. The curvatures in different circles vary inversely as the diameters.

Let PQV, pqv be two circles, draw the diameters PV, pv, and the tangents PR, pr. Take $PR = pr$, and draw the subtenses QR, qr parallel to the diameters, and QN, qn parallel to the tangents;



$$\text{then } \frac{QR}{qr} = \frac{PN}{pn} = \frac{QN^2}{NV} \div \frac{qn^2}{nv} = \frac{nv}{NV},$$

$$\therefore \frac{\text{curvature at } P}{\text{curvature at } p} = \text{limit}\,\frac{QR}{qr}$$

$$= \text{limit}\,\frac{nv}{NV}$$

$$= \frac{pv}{PV}$$

or the curvature $\propto \dfrac{1}{\text{diameter}}$.

Cor. Hence in the same circle the curvature is the same at every point.

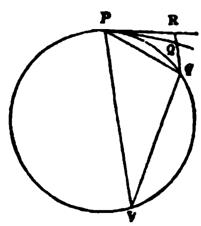
From this property of the circle, and also because by varying the diameter it may be made to have any curvature we please, the circle is made use of to measure the curvature at any proposed points of other curves.

Def. The *circle of curvature* at any point of a curve is that circle which has the same tangent and curvature as the curve has at that point, the curvatures being in the same direction.

Hence if QqR be a common subtense to the curve PQ and the circle Pq, and limit $\dfrac{QR}{qR} = 1$, Pq will be the circle of curvature at P.

The radius, diameter, and chord of the circle of curvature are generally called the radius, diameter, and chord of curvature.

Prop. III. If PqV be the circle of curvature at any point P, and PV a chord drawn in any given direction, then



$$PV = \text{limit}\ \frac{(\text{arc})^2}{\text{subtense parallel to the chord}}.$$

Take PQ an arc of the curve, through Q draw the subtense RQq parallel to PV, and join Pq, qV; then since the triangles PRq, PqV are evidently similar,

$$PV = \frac{Pq^2}{qR}.$$

Now this being true whatever be the magnitude of PQ, will be true when RQr moves parallel to itself up to P, in which case $Pq = PQ$ ultimately, and $qR = QR$ ultimately,

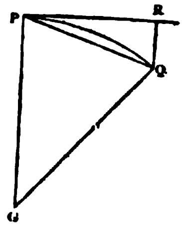
$$\therefore PV = \text{limit} \frac{Pq^2}{qR}$$

$$= \text{limit} \frac{(\text{arc } PQ)^2}{QR}.$$

Cor. Hence the diameter of curvature

$$= \text{limit} \frac{(\text{arc})^2}{\text{subtense perpendicular to the tangent}}.$$

Prop. IV. If in the curve PQ, PG and QG, drawn perpendicular to the tangent PR and the chord PQ respectively, intersect in G, then when Q moves up to P, the limit of PG is the diameter of curvature at P.



Draw the perpendicular subtense QR,

Then by similar triangles PGQ, PQR,

$$PG = \frac{PQ^2}{QR}:$$

$$\therefore \text{limit } PG = \text{limit} \frac{PQ^2}{QR}$$

$$= \text{limit} \frac{(\text{arc } PQ)^2}{QR}$$

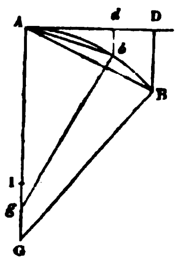
$$= \text{diameter of curvature at } P.$$

Def. The curvature of a curve at any point is said to be finite, when the diameter of curvature at that point is finite.

LEMMA XI.

In curves of finite curvature the limiting ratio of the subtenses equals that of the squares of the conterminous arcs.

Let AbB be the curve having a finite curvature at A;



First, Let the subtenses bd, BD be perpendicular to the tangent at A. Draw bg BG at right angles to the chords Ab, AB, and let them meet AgG, which is drawn at right angles to the tangent AD, in the points g and G.

Then as b and B move up to A, g and G move up to I, the extremity of the diameter of curvature of A, as their limit. (Prop. IV.)

Now by similar triangles,

$$BD = \left(\frac{AB^2}{AG}\right), \quad bd = \frac{Ab^2}{Ag}$$

$$\therefore \; BD : bd = \frac{AB^2}{AG} : \frac{Ab^2}{Ag},$$

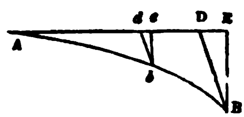
$$\therefore \; \text{L. R. } BD : bd = \text{L. R. } \frac{AB^2}{AG} : \frac{Ab^2}{Ag}$$

$$= \text{L. R. } AB^2 : Ab^2,$$

(since AG, Ag are ultimately equal to AI)

$$= \text{L. R. (arc } AB)^2 : (\text{arc } Ab)^2.$$

Secondly, Let the subtenses be inclined at any equal angles to the tangent. Draw BE, be perpendicular to the tangent: then by similar triangles,



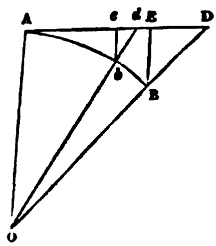
$$BD : BE = bd : be,$$

alternando $BD : bd = BE : be$;

$$\therefore \text{ L. R. } BD : bd = \text{L. R. } BE : be$$

$$= \text{L. R. (arc } AB)^2 : (\text{arc } Ab)^2.$$

Thirdly, Let the subtenses, inclined at unequal angles to the tangent, converge to a point, and revolve round that point fixed, or approach to A according to any other given law.



Let O be the point in which DB, db meet when produced; draw BE, be always parallel to AO; then since the angles at D and d are always finite, AO must always be finite, and L. R. $DO : AO$ will be a ratio of equality, as also L. R. $dO : AO$

$$\left.\begin{array}{ll}\text{But } & BD : BE = DO : AO \\ \text{and } & bd : be = dO : AO\end{array}\right\}, \text{ always and therefore ultimately;}$$

$$\therefore \text{ L. R. } BD : BE = \text{L. R. } bd : be;$$

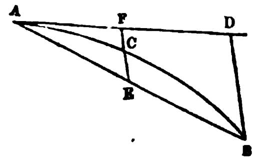
$$\therefore \text{ L. R. } BD : bd = \text{L. R. } BE : be$$

$$= \text{L. R. (arc } AB)^2 : (\text{arc } Ab)^2.$$

Cor. 1. Hence by Lemma vii. the limiting ratio of the subtenses will equal that of the squares of the arcs, chords, and tangents.

Theorem. *The subtense of an arc is ultimately equal to four times the parallel sagitta.*

Def. The sagitta of an arc is a line drawn at a finite angle to the chord from its middle point to meet the arc.



Let BD be a subtense of the arc AB, EC the sagitta parallel to it, bisecting the chord in E, and produced to meet the tangent in F.

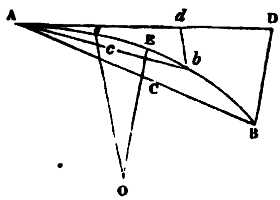
Then by similar triangles,

$$AF = \tfrac{1}{2} AD, \text{ and } EF = \tfrac{1}{2} BD.$$

Also by the Lemma,

$$\text{L. R. } CF : BD = \text{L. R. } AF^2 : AD^2$$
$$= 1 : 4$$
$$\therefore \text{L. R. } CE : BD = 1 : 4.$$

COR. 2. The limiting ratio of the sagittæ, which bisect the chords and converge to a given point, equals that of the squares of the arcs, chords, and tangents.



Let EC, ec be the sagittæ of the arcs AEB, Aeb, bisecting the chords AB, Ab in C, c; draw the subtenses BD, bd respectively parallel to them;

then $\text{L. R. } EC : BD = 1 : 4$

$$= \text{L. R. } ec : bd;$$

$$\therefore \text{L. R. } EC : ec = \text{L. R. } BD : bd;$$
$$= \text{L. R. } (\text{arc } AB)^2 : (\text{arc } Ab)^2$$
$$= \text{L. R. } (\text{chord } AB)^2 : (\text{chord } Ab)^2$$
$$= \text{L. R. } (\text{tangent } AD)^2 : (\text{tangent } Ad)^2.$$

COR. 3. Hence if a body describe the arcs AB, Ab with any given velocity, the limiting ratio of the sagittæ will be that of the squares of the times, in which they are described.

Cor. 4. If the subtenses DB, db be perpendicular to the tangent, as in the first case of the Lemma,

$$\triangle ADB : \triangle Adb = AD \,.\, DB : Ad \,.\, db;$$

$$\begin{aligned} \therefore \text{L. R. } \triangle ADB : \triangle Adb &= \text{L. R. } AD \,.\, DB : Ad \,.\, db \\ &= \text{L. R. } AD^3 : Ad^3 \\ \text{or } &= \text{L. R. } DB^{\frac{3}{2}} : db^{\frac{3}{2}}. \end{aligned}$$

Cor. 5. Since L. R. $DB : db =$ L. R. $AD^2 : ad^2$, the limiting form to which every curve of finite curuature approximates is that of the common parabola.

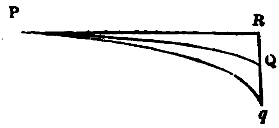
Hence also,

$$\begin{aligned} \text{L. R. area } ADB : \text{area } Adb &= \text{L. R. } \tfrac{1}{3} AD \,.\, DB : \tfrac{1}{3} Ad \,.\, db \\ &= \text{L. R. } AD^3 : Ad^3 \\ \text{or } &= \text{L. R. } DB^{\frac{3}{2}} : db^{\frac{3}{2}}. \end{aligned}$$

Scholium to Lemma XI.

It was proved in the Lemma that if the curvature be finite, the subtense varies ultimately as the square of the conterminous arc; conversely,

If the subtense vary ultimately as the square of the arc, the curvature is finite, and if it vary according to any other power of the arc, the curvature is infinitely great or infinitely small.



Let PQ and Pq be arcs of a curve and circle, having a common tangent PR, and let RQq be a common subtense.

Since in the circle $qR \propto$ ult. PR^2, let $qR = a \,.\, PR^2$ ultimately,

and suppose that $QR \propto$ ult. PR^n and $= b \,.\, PR^n$ ultimately,

$$\therefore \frac{\text{curvature of } PQ}{\text{curvature of } Pq} = \text{limit } \frac{QR}{qR} = \frac{b}{a} \,.\, \text{limit } PR^{n-2}.$$

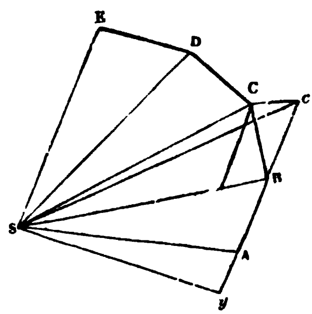
If $n = 2$, the curvature of the curve PQ bears a finite ratio to that of the circle, and is therefore finite. If n be greater than 2, limit $PR^{n-2} = 0$, and therefore the curvature of PQ is infinitely small compared with that of Pq, and the curve will lie between Pq and the tangent. If n be less than 2, limit $PR^{n-2} = \infty$, and therefore the curvature of PQ is infinitely great, and the curve will lie below Pq.

Cor. Since an infinite number of values may be given to n, to each of which there will be a corresponding curve, an infinite number of curves may be described between Pq and the tangent, corresponding to values of n greater than 2, and an infinite number below Pq, corresponding to values of n less than 2.

SECTION II.

ON THE MOTION OF A BODY, CONSIDERED AS A POINT, MOVING IN A NONRESISTING MEDIUM, AND ATTRACTED TO A SINGLE FIXED CENTER OF FORCE.

PROP. I. *If a body move in any orbit about a fixed center of force, the areas, described by lines drawn from the center to the body, lie in one plane, and are proportional to the times of describing them.*



Let S be the center of force; and suppose a body unattracted by the force in S to describe the straight line AB with a uniform velocity in a given time (T). Then if suffered to proceed, it would move on uniformly in the direction of AB produced, and describe $Bc = AB$ in the next interval (T); but at B suppose an instantaneous impulse communicated to it in direction BS, which causes it to move in direction BC; draw cC parallel to BS, then by the principles of Mechanics, the body at the end of the second interval will be found at C. Join SA, SB, Sc, SC. Since cC is parallel to BS, the triangle $SBC = SBc = SAB$, since $Bc = AB$; and these triangles are in the same plane, as no force has acted to draw the body out of the plane SAB. Similarly, if impulses be communicated at the end of every interval of T'', in directions tending always

to S, causing the body to describe CD, DE, &c. in the third, fourth, &c. intervals, the triangles SAB, SBC, SCD, &c. will be all equal, and will lie in the same plane; and their bases AB, BC, CD, &c. are described in equal times, therefore the area of any number of these triangles or the polygon $SABCDE$ varies as the time of describing it. Now let the number of intervals be increased, and the magnitude of each diminished indefinitely, then the polygon approximates to a curvilinear area, and the sum of the impulses to a continued force always tending to S, as their limits; and what was proved of those quantities is true of their limits, and therefore the curvilinear area described in any time is proportional to the time.

Obs. The area, described by the line joining S and the body, is frequently called the area described by the body round S.

Cor. 1. If V be the velocity of the body at A, and p the perpendicular from S upon the tangent at that point, the area described in $t'' = \frac{1}{2}p \cdot t \cdot v$.

Draw Sy perpendicular to AB; then since AB is ultimately the tangent at A, limit of $Sy = p$. Also if t be divided into n equal intervals, and AB be the space described in the first interval, the force in S being supposed, as in the Prop., not to act, $AB = \frac{t}{n} . v$.

Hence, polygonal area described in $t'' = n$. triangle SAB

$$= n \cdot \tfrac{1}{2} Sy . \frac{t}{n} . v$$

$$= \tfrac{1}{2} Sy . t . v;$$

and the same is true in the limit,

$\therefore$ curvilinear area described in $t'' = \frac{1}{2}p \cdot t \cdot v$.

Cor. 2. Hence the time of describing any part of the orbit

$$= \frac{2}{p \cdot v} \text{. area described.}$$

Cor. 3. If $t = 1$, area described in $1'' = \frac{1}{2} p \cdot v$.

Hence in different orbits, the velocity at any point

$$\propto \frac{\text{area described in } 1''}{\text{perpendicular from } S \text{ upon the tangent}},$$

and in the same orbit, the velocity

$$\propto \frac{1}{\text{perpendicular upon the tangent}}.$$

Prop. II. *If a body, moving in a curve, describe in one plane areas proportional to the times by lines drawn from the body to any point, the body is acted on by centripetal forces all tending to that point.* (Vide Fig. Prop. 1.)

Let S be the point, about which areas proportional to the times are described; and suppose as in Prop. 1. that a body, unattracted by the force in S, describes the straight line AB in a given time T.

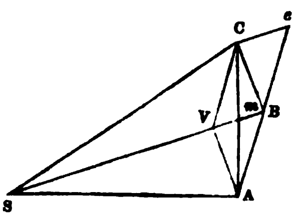
In AB produced take $Bc = AB$; then if suffered to proceed, the body would be at c at end of the second interval of T''. But at B suppose an impulse communicated, which causes it to describe BC in the second interval, such that the triangle SBC is equal to and in the same plane with the triangle SAB. Join cC, Sc.

Then the triangle $SBC = SAB = SBc$, therefore cC is parallel to BS, and therefore by the principles of Mechanics the impulse communicated at B tends to S. Similarly if D, E, &c. be the places of the body at the ends of the third, fourth, &c. intervals of T'', so that the triangles SAB, SBC, SCD, &c. are all equal, all the impulses communicated may be shewn to tend to S.

Now suppose the number of intervals increased, and the magnitude of each diminished indefinitely, then the limit of the

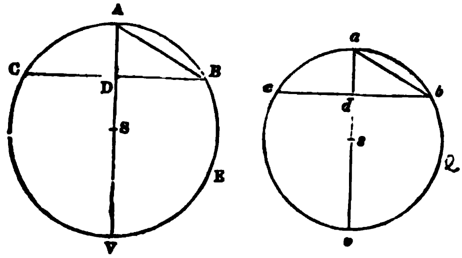
polygon is the curvilinear area, and that of the sum of the impulses a continued force tending to S; and the above reasoning still holds in the limit, therefore the body is acted on by a continued force tending to S.

Cor. Draw CV parallel to AB meeting SB in V, and join AV. Then $CV = Bc = AB$, $\therefore AV$ is equal and parallel to CB, or $ABCV$ is a parallelogram. Draw the diagonal CA bisecting BV in m.



Now suppose $S'A'B'C'D'$ to be another orbit, in which the chords $A'B'$, $B'C'$ are described in the same time as either of the chords AB or BC; and let the same construction be made as in the former orbit, then impulse at B : impulse at $B' = cC : c'C' = Bm : B'm'$ and therefore force at B : force at $B' =$ L. R. $Bm : B'm'$; or the centripetal forces in different orbits are in the limiting ratio of the sagittæ of arcs described in equal times, which pass through the centers of force.

Prop. III. *The centripetal forces, by which bodies describe different circles with uniform velocities, tend to the centers of the circles, and are as the squares of the arcs, described in the same time, divided by the radii.*



Since in each circle the motion is uniform, the arcs described are proportional to the times. But the sectors, i.e. the areas described about the centers of the circles, are as the arcs on which they stand, and are therefore proportional to the times; hence (Prop. II.) the forces tend to the centers of the circles.

Again let CAB, cab be arcs described in the same time in the circles, whose centers are S, s, and let A, a be their middle points; join AB, ab, and draw the diameters ASV, asv cutting the chords CB, cb in D, d; then (Prop. II. Cor.)

$$\text{Force at } A : \text{force at } a = \text{L. R. } AD : ad$$

$$= \text{L. R. } \frac{(\text{chord } AB)^2}{AV} : \frac{(\text{chord } ab)^2}{av}$$

$$= \text{L. R. } \frac{(\text{arc } AB)^2}{AS} : \frac{(\text{arc } ab)^2}{as}.$$

Take AE, ae any other arcs described in equal times;

$$\text{then } AE : ae = AB : ab,$$

and this being true whatever be the magnitudes of AB, ab will be true when they are diminished indefinitely,

$$\therefore AE : ae = \text{L. R. } AB : ab,$$

and therefore force at A : force at $a = \dfrac{AE^2}{AS} : \dfrac{ae^2}{as}$.

Cor. 1. Since AE = velocity × time, if V = velocity of the body, R = radius of the circle, and the time be given,

$$F \propto \frac{V^2}{R}.$$

Cor. 2. Let P equal the periodic time, then since $s = tv$,

$$2\pi R = P.V; \quad \therefore F \propto \frac{R^2}{P^2 . R} \propto \frac{R}{P^2}.$$

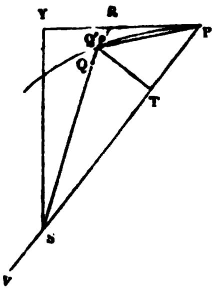
Cor. 3. If P be given, $F \propto R$. If $P \propto R^{\frac{3}{2}}$, $F \propto \dfrac{1}{R^2}$,

and generally if $P \propto R^n$, $F \propto \dfrac{1}{R^{2n-1}}$.

R. C.

Prop. VI. *A body moving round a fixed center of force S, describes the arc PQ in T''; if F be the central force at P, and QR a subtense parallel to SP, when PQ and T are diminished indefinitely,*

$$F = 2 \text{ limit } \frac{QR}{T^2}.$$



The motion of the body on leaving P is compounded of two motions, one uniform in direction of the tangent PR, the other variable, arising from the action of S, by which the body is deflected from the tangent, and which tends continually, as Q approximates to P, to become parallel to PS. Let RQ' be the space, through which the force F, continued constant and always parallel to PS, would draw the body in T'',

$$\text{then } F = 2\,\frac{Q'R}{T^2};$$

and this is true when T is diminished indefinitely, in which case $Q'R = QR$, and therefore

$$F = 2 \text{ limit } \frac{QR}{T^2}.^*$$

Cor. 1. Draw QT perpendicular to SP, and join SQ, QP; let $h = 2$ area described in $1''$,

area of $\triangle$ $PSQ = \frac{1}{2}$

$$\text{then } \frac{T''}{1''} = \frac{2 \text{ area } PSQ}{h},$$

$\dfrac{\text{area describ}^d \text{ in } T''}{1''} = \dfrac{T''}{1''}$

---

* The above expression for the force being obtained independently of the preceding propositions, it is not necessary that the areas described should be proportional to the times. It is therefore true in orbits described round several centers of force, in which case the expression represents the magnitude of the resultant of all the forces acting on the body at the point P. It is clear, however, that the equable description of areas is supposed to be preserved in the three succeeding corollaries. The result in Cor. 4, is general, and might easily be obtained from Cor. 3, in the particular case of the areas being described equably, by substituting for h its value $Sy \, . \, V$, obtained in Cor. 2. Prop. 1.

$$\therefore\ F = 2\,\text{limit}\,\frac{QR}{T^2} = 2\,\text{limit}\,\frac{QR\,.\,h^2}{4(\text{area } PSQ)^2}$$

$$= 2\,\frac{h^2}{4}\,\text{limit}\,\frac{QR}{(\Delta PSQ)^2} = 2\,\frac{h^2}{4}\,.\,\text{limit}\,\frac{QR}{\frac{1}{4}QT^2\,.\,SP^2}$$

$$= \frac{2h^2}{SP^2}\,.\,\text{limit}\,\frac{QR}{QT^2}.$$

Cor. 2. Draw SY perpendicular to the tangent PR, then since the angle QPR ultimately vanishes, the triangles QPT, SPY are ultimately similar;

$$\therefore\ \text{limit}\,\frac{QT}{PQ} = \frac{SY}{SP},$$

$$\therefore\ \text{limit}\,\frac{QR}{QT^2} = \frac{SP^2}{SY^2}\,\text{limit}\,\frac{QR}{PQ^2},$$

$$\therefore\ F = \frac{2h^2}{SY^2}\,\text{limit}\,\frac{QR}{PQ^2}.$$

Cor. 3. If PV be the chord of curvature at P through S,

$$PV = \text{limit}\,\frac{PQ^2}{QR},\quad \therefore\ F = \frac{2h^2}{SY^2\,.\,PV}.$$

Obs. If A = the area described in P'', $h = \dfrac{2A}{P}$, which value may be substituted for h in the above expressions for the force.

Cor. 4. *The space, through which a body must descend from rest by the action of the force at P continued constant, in order to acquire the velocity at P, is $\frac{1}{4}$th of the chord of curvature PV.*

Since limit $\dfrac{PR}{PQ} = 1$, $F = 2\,\text{limit}\,\dfrac{QR}{T^2} = 2\,\text{limit}\,\dfrac{QR}{PQ^2}\,.\,\left(\dfrac{PR}{T}\right)^2.$

Now limit $\dfrac{QR}{PQ^2} = \dfrac{1}{PV}$, and limit $\dfrac{PR}{T} =$ velocity at $P = V$;

$$\therefore \; F = \frac{2V^2}{PV}, \text{ and therefore } V^2 = F \cdot \frac{PV}{2}.$$

Let $S =$ space due to V by the action of F continued constant,

$$\text{then, } V^2 = 2FS,$$

hence equating this to the above expression for V^2, we have

$$S = \tfrac{1}{4} PV.$$

Cor. 5. To find the velocity and periodic time of a body revolving in a circle and acted on by a centripetal force tending to the center of the circle.

$$\text{Here } PV = \text{the diameter} = 2R, \quad \therefore v = \sqrt{F \cdot R};$$

$$\text{Also } P = \frac{\text{circumference}}{\text{velocity}} = \frac{2\pi R}{\sqrt{F \cdot R}} = 2\pi \sqrt{\frac{R}{F}}.$$

Lemma. If P, p be points similarly situated in similar orbits described round S, s centers of force also similarly situated, and PV, pv be chords of curvature drawn through the centers of force,

$$SP : sp = PV : pv.$$

Take (Fig. Cor. 6.) PQ, pq similar arcs, and draw the subtenses QR, qr parallel to SP, sp: then by similar figures $SPRQ$, $sprq$,

$$SP^2 : sp^2 = PQ^2 : pq^2,$$

$$SP : sp = QR : qr,$$

$$\therefore SP : sp = \frac{PQ^2}{QR} : \frac{pq^2}{qr}$$

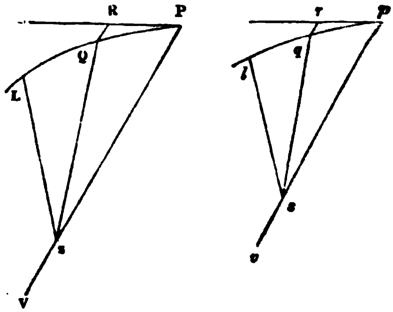
$$= \text{L. R.} \frac{PQ^2}{QR} : \frac{pq^2}{qr}$$

$$= PV : pv.$$

omit.– Cor. 6. If V, v be the velocities at P, p, points similarly situated in similar orbits, described round S, s centers of force, also similarly situated,

$$\text{Force at } P\,(F) : \text{force at } p\,(f) = \frac{V^2}{SP} : \frac{v^2}{sp}.$$

Let PQ, pq be arcs described in equal times, QR, qr subtenses parallel to SP, sp, and PV, pv chords of curvature at P, p through S, s.



Then since the times are equal,

$$F : f = \text{L.R. } QR : qr$$

$$= \text{L.R. } \frac{PQ^2}{PV} : \frac{pq^2}{pv},$$

$$\text{also } V : v = \text{L.R. } \frac{PQ}{T} : \frac{pq}{T}$$

$$= \text{L.R. } PQ : pq,$$

and since P and p are points similarly situated in similar orbits,

$$SP : sp = PV : pv,$$

by the Lemma in the preceding page,

$$\therefore F : f = \frac{V^2}{SP} : \frac{v^2}{sp}.$$

Cor. 7. If similar arcs of similar orbits be described in times T, t round S, s, centers of force similarly situated, (Fig. Cor. 6.)

$$F : f = \frac{SP}{T^2} : \frac{sp}{t^2}.$$

Let PL, pl be similar arcs described in times T, t, and take PQ, pq other similar arcs described in times P, p; QR, qr subtenses parallel to SP, sp; then

$$F : f = \text{L. R.}\ \frac{QR}{P^2} : \frac{qr}{p^2},$$

join SQ, SL, sq, sl.

Then $T : P =$ area PSL : area PSQ (Prop. I.)

$=$ area psl : area psq, by similar figures,

$= t : p$, (Prop. I.)

$$\therefore\ T : t = P : p;$$

and this, being always true, will be true when P and p are diminished indefinitely,

$$\therefore\ T : t = \text{L. R.}\ P : p,$$

and by similar figures,

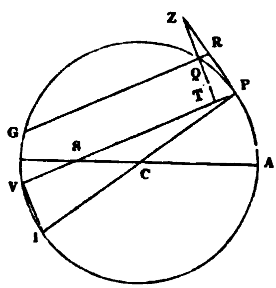
$SP : sp = QR : qr$ always and therefore in the limit;

$$\therefore\ F : f = \frac{SP}{T^2} : \frac{sp}{t^2}.$$

Q

PROP. VII. *A body revolves in the circumference of a circle, to find the law of force by which it is attracted to a given point.*

Let PAV be the circumference of the circle, and S the center of force; PQ an arc, QR a subtense parallel to SP, QT perpendicular to SP. Let RQ, and PS, produced if necessary, meet the circumference in G, V; draw the diameter PI, join IV, and produce TQ, PR to meet in Z. The triangles PTZ, PVI are evidently similar.



$$\text{Hence}\ \frac{QR \cdot RG}{QT^2} = \frac{RP^2}{QT^2}\ (\text{Euc. III. 36.}) = \frac{ZP^2}{ZT^2} = \frac{PI^2}{PV^2}.$$

Now let Q move up to P,

$$\text{then limit } \frac{QR}{QT^2} = \text{limit } \frac{PI^2}{PV^2 \, . \, RG}$$

$$= \frac{PI^2}{PV^3}, \text{ since limit } RG = PV.$$

$$\therefore \; F = \frac{2h^2}{SP^2} \, . \, \text{limit } \frac{QR}{QT^2}$$

$$= \frac{2h^2}{SP^2} \cdot \frac{PI^2}{PV^3} = \frac{8h^2R^2}{SP^2 \, . \, PV^3},$$

if R = radius of the circle.

Let μ represent that part of the expression for F, which in the same orbit is invariable; then in this case,

$$\mu = 8h^2R^2.$$

$$\text{Hence } F = \frac{\mu}{SP^2 \, . \, PV^3},$$

and therefore in the same circle $\propto \dfrac{1}{SP^2 \, . \, PV^3}$.

omit

Cor. 1. To find the velocity at any point.

$$V^2 = F \, . \, \frac{PV}{2} = \frac{4h^2 \, . \, R^2}{SP^2 \, . \, PV^2}, \text{ or } = \frac{\mu}{2 \, . \, SP^2 \, . \, PV^2};$$

$$\therefore \; V = \frac{2hR}{SP \, . \, PV}, \text{ or } = \sqrt{\frac{\mu}{2}} \cdot \frac{1}{SP \, . \, PV}.$$

Obs. The quantity (μ) here introduced is that part of the general expression for the centripetal force in any orbit, which is invariable for all points in that orbit, and may always be determined, if the actual force at any given point be known. The force, by which a body is retained in a given curve, is in most cases undergoing a continual change in magnitude; but its magnitude at any given point is to be

estimated by the effect it would produce, that is, by the velocity it would generate in a unit of time from rest, supposing it to remain constant for that time. Hence, if a second and a foot be the units of time and space, the magnitude of the centripetal force at any point is represented by twice the number of feet, which it would cause a body to describe from rest in 1″; if for instance, it draws a body from rest through 10 feet in 1″, its magnitude will be 20, and it will be to the force of gravity in the ratio of 20 : $32\frac{1}{5}$ or of 100 : 161. Suppose then in the preceding proposition, that the force at A, the extremity of the diameter through S, would if continued constant draw a body through (f) feet in 1″;

$$\text{then } \frac{\mu}{SA^2 \,.\, (2R)^3} = 2f;$$

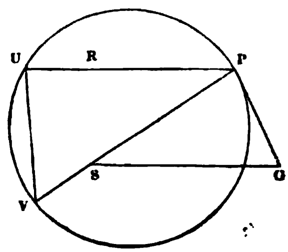
$$\therefore \mu = 2f \,.\, SA^2, \ (2R)^3.$$

Cor. 2. Let S be in the circumference, then $PV = SP$.

$$\text{Hence } F = \frac{8h^2 R^2}{SP^5}, \quad \text{or} = \frac{\mu}{SP^5}; \quad \text{and therefore,} \propto \frac{1}{SP^5},$$

$$V = \frac{2hR}{SP^2}, \quad \text{or} = \sqrt{\frac{\mu}{2}} \,.\, \frac{1}{SP^2}.$$

Cor. 3. To compare the forces, by which a body, attracted separately to two centers of force, may describe the same circle in the same periodic time.



Let R and S be the two centers of force; produce PR, PS if necessary to meet the circumference in U, V; draw SG parallel to RP to meet the tangent at P in G, and join UV; then the triangles SPG, PUV are evidently similar,

$$\therefore \frac{SG}{SP} = \frac{PV}{PU}, \quad \text{or } SG = \frac{SP \,.\, PV}{PU}.$$

Also since the periodic time is the same, h, which

$$= \frac{2 \text{ area of circle}}{\text{periodic time}},$$

is the same for both centers, hence

$$F \text{ to } R : F \text{ to } S = \frac{1}{RP^2 \,.\, PU^3} : \frac{1}{SP^2 \,.\, PV^3}$$

$$= \frac{SP^3 \,.\, PV^3}{PU^3} : RP^2 \,.\, SP$$

$$= SG^3 : RP^2 \,.\, SP.$$

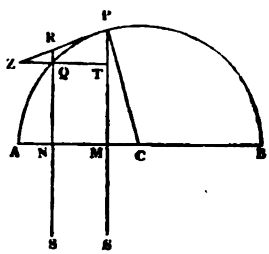
Cor. 4. What has been proved in the last corollary in the case of the circle is true of any orbit described round two centers of force separately in the same periodic time. For if PUV be the circle of curvature at P, the expression for F, viz. 2 limit $\frac{QR}{QT^2}$, is the same in the curve and circle, and therefore what has been proved in the one case is true in the other. Hence generally in any orbit described in the same time round two centers of force,

$$F \text{ to } R : F \text{ to } S = SG^3 : RP^2 \,.\, SP.$$

If the periodic times are not the same,

$$F \text{ to } R : F \text{ to } S = \frac{SG^3}{P^2 \text{ round } R} : \frac{RP^2 \,.\, SP}{P^2 \text{ round } S}.$$

Prop. VIII. *To find the law of force by which a body may describe a semicircle, the center of force being so distant, that all lines drawn from it to the body may be considered parallel.*



Let PQ be an arc of the semicircle, C the center; draw PS, QS parallel to each other towards the center of force, and CM perpendicular to PS; then CM produced both ways will determine the semicircle described. Draw QT perpendicular, and QR parallel to SP, and produce PR, TQ to meet in Z; join CP. The triangles PZT, CPM are evidently similar.

$$\therefore \frac{QR \,.\, (RN + QN)}{QT^2} = \frac{RP^2}{QT^2} = \frac{ZP^2}{ZT^2} = \frac{CP^2}{PM^2};$$

$$\therefore \text{limit } \frac{QR}{QT^2} = \frac{CP^2}{2PM^3}, \quad \text{since limit } (RN + QN) = 2PM;$$

$$\therefore F = \frac{2h^2}{SP^2} \,.\, \text{limit } \frac{QR}{QT^2}$$

$$= \frac{h^2 \,.\, CP^2}{SP^2 \,.\, PM^3}, \text{ or } = \frac{\mu}{PM^3}, \text{ and } \therefore \propto \frac{1}{PM^3}.$$

Cor. To find the velocity at any point.

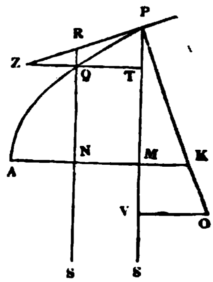
$$V^2 = F \,.\, \frac{PV}{2} = \frac{h^2 \,.\, CP^2}{SP^2 \,.\, PM^3} \,.\, PM$$

$$= \frac{h^2 \,.\, CP^2}{SP^2 \,.\, PM^2},$$

$$\therefore V = \frac{h \,.\, CP}{SP \,.\, PM}, \text{ or } = \frac{\sqrt{\mu}}{PM}.$$

Scholium to Prop. VIII.

If AQP be any conic section, it may be described by the action of a force tending to a point at an infinite distance, and varying inversely as the cube of the ordinate.



Let PO, the diameter of curvature at P, cut the axis of the conic section in K; draw OV perpendicular to PS, then PV is the chord of curvature at P in direction of the force; and complete the construction as in the proposition.

By similar triangles ZPT, PMK,

$$\frac{QT^2}{QR} : \frac{RP^2}{QR} = ZT^2 : ZP^2 = PM^2 : PK^2,$$

and this being true always will be true when Q moves up to P,

$$\therefore \text{ L. R. } \frac{QT^2}{QR} : \frac{RP^2}{QR} = PM^2 : PK^2,$$

$$\text{and } PV : PO = PM : PK,$$

$$\therefore \text{ since limit } \frac{RP^2}{QR} = \text{limit} \frac{PQ^2}{QR} = PV,$$

$$\text{limit} \frac{QT^2}{QR} : PO = PM^3 : PK^3.$$

Now (Appendix, Arts. 4, 5.) in all conic sections, the diameter of curvature $= \dfrac{8}{L^2} \cdot PK^3$,

$$\therefore \text{ limit} \frac{QT^2}{QR} = \frac{8 PM^3}{L^2},$$

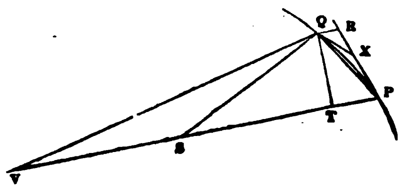
$$\text{and } \therefore F = \frac{h^2 \cdot L^2}{4 SP^2 \cdot PM^3} \propto \frac{1}{PM^3}.$$

Look at this.

Cancel

Prop. IX. *To find the law of force tending to the pole, by which a body may describe an equiangular spiral.*

Def. An equiangular spiral is a spiral, which cuts every radius vector at the same given angle.



Let PQ be an arc of the spiral, S the center of force in the pole. QR a subtense parallel to SP, QT perpendicular to SP, and let the constant angle SPR, which the curve makes with the radius, $= \alpha$. Let PV be the chord of curvature through S, and join PQ, QV;

$$\text{then } \frac{QT^2}{QR} = \frac{PR^2}{QR} \sin^2 \alpha;$$

$$\therefore \text{ limit } \frac{QT^2}{QR} = \text{limit } \frac{PR^2}{QR} \sin^2 \alpha = \text{limit } \frac{PQ^2}{QR} \sin^2 \alpha = PV \sin^2 \alpha.$$

Let the tangent at Q intersect PR in X. Then since SP, SQ make equal angles with the tangents at P, Q, the angles SPX, SQX are equal to two right angles, therefore the angle PSQ = angle QXR. Also since V is a point in the circumference of the circle of curvature, the angles XPQ, XQP are each ultimately equal to QVP. Hence the angle QXR, and therefore the angle QSP is ultimately double of the angle QVS, therefore $\angle\, SQV$ is ultimately equal to $\angle\, SVQ$, or $SV = SQ$ ultimately $= SP$. Hence $PV = 2SP$,

$$\text{and } \therefore\ F = \frac{2h^2}{SP^3} \text{ limit } \frac{QR}{QT^2}$$

$$= \frac{2h^2}{SP^2} \cdot \frac{1}{2SP\sin^2\alpha}$$

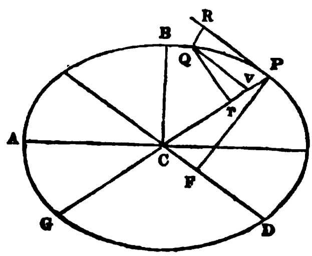
$$= \frac{h^2}{\sin^2\alpha} \cdot \frac{1}{SP^3}, \text{ or } = \frac{\mu}{SP^3}, \text{ and } \therefore \propto \frac{1}{SP^3}.$$

Cor. To find the velocity at any point.

$$V^2 = F \cdot \frac{PV}{2} = \frac{h^2}{\sin^2\alpha} \cdot \frac{1}{SP^2}, \text{ or } \frac{\mu}{SP^2};$$

$$\therefore V = \frac{h}{\sin\alpha} \cdot \frac{1}{SP}, \text{ or } \frac{\sqrt{\mu}}{SP}.$$

Prop. X. *A body describes an ellipse round a center of force in the center of the ellipse, to find the law of force.*



Let PQ be an arc of the ellipse, C the center, QR a subtense parallel to CP; AC, BC the semi-axes major and minor; QV parallel to PR; QT, PF perpendicular to CP and the semi-conjugate CD respectively, produce PC to meet the ellipse again in G; then the triangles QVT, PCF are evidently similar.

$$\text{Now } \frac{PV \cdot VG}{QV^2} = \frac{CP^2}{CD^2},$$

$$\text{and } \frac{QV^2}{QT^2} = \frac{CP^2}{PF^2};$$

$$\therefore \frac{PV \cdot VG}{QT^2} = \frac{CP^4}{PF^2 \cdot CD^2} = \frac{CP^4}{AC^2 \cdot BC^2};$$

$$\therefore \text{ limit } \frac{QR}{QT^2} = \text{limit } \frac{PV}{QT^2} = \frac{CP^4}{AC^2 \cdot BC^2 \cdot 2CP},$$

(since limit $VG = 2CP$)

$$= \frac{CP^3}{2AC^2 \cdot BC^2};$$

$$\therefore F = \frac{2h^2}{CP^3} \text{limit } \frac{QR}{QT^2} = \frac{h^2}{AC^2 \cdot BC^2} \cdot CP,$$

or $= \mu \cdot CP$, and therefore $\propto CP$

Cor. 1. To find the velocity at any point.

$$V^2 = \tfrac{1}{2} F \cdot PV = \tfrac{1}{2} \frac{h^2}{AC^2 \cdot BC^2} CP \cdot \frac{2CD^2}{CP}, \text{ since } PV = \frac{2CD^2}{CP}$$

$$= \frac{h^2}{AC^2 \cdot BC^2} CD^2;$$

$$\therefore V = \frac{h}{AC \cdot BC} \cdot CD, \text{ or } \sqrt{\mu} \cdot CD.$$

Cor. 2. To find the periodic time.

$$\text{Since } \mu = \frac{h^2}{AC^2 \cdot BC^2}, \; h = AC \cdot BC \cdot \sqrt{\mu};$$

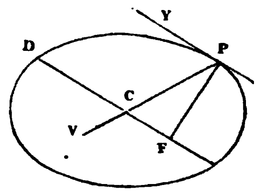
also the area of the ellipse $= \pi AC \cdot BC$;

$$\therefore P = \frac{2 \text{ area of ellipse}}{h}$$

$$= \frac{2\pi}{\sqrt{\mu}}.$$

Hence the periodic times in all ellipses round the same center of force in the center are equal.

Cor. 3. *If a body be projected in a direction making any angle with its distance from a fixed point, and be attracted to that point by a force varying as the distance, it will describe an ellipse, whose center is the center of force.*



Let C be the center of force, P the point from which the body is projected in direction PY, V the velocity, and F the force at P.

Then space (s) due to the velocity at $P = \dfrac{V^2}{2F}$. In PC, produced if necessary, take $PV = 4s$, and draw CD parallel to PY and $= \sqrt{\tfrac{1}{2} CP \cdot PV}$. With CP, CD as semi-conjugate diameters describe an ellipse, and suppose a body revolving in it to come to P; then it is moving in the direction of the tangent at P, that is, in a line parallel to CD or in direction PY. Also space due to velocity at $P = \frac{1}{4}$ chord of curvature at P,

$$= \tfrac{1}{4} \cdot \frac{2\,CD^2}{CP} = \tfrac{1}{4} PV = s.$$

The force, distance, and law of force are the same also in both cases; hence the two bodies are under the same circumstances at P, and will therefore describe the same orbit; that is, the projected body will describe an ellipse, whose center is C.

If CPY be a right angle, and $s = \frac{1}{2} PC$, the orbit described will be a circle.

Cor. 4. To compare the velocity at P with the velocity in a circle, radius $= CP$, described round the same center of force.

$$V \text{ in ellipse} = \sqrt{\mu} \cdot CD.$$

$$V \text{ in circle (radius} = CP) = \sqrt{F \cdot CP}, \text{ (Prop. VI. Cor. 5.)}$$

$$= \sqrt{\mu} \cdot CP;$$

$$\therefore\ V \text{ in ellipse} : V \text{ in circle (rad.} = CP) = CD : CP.$$

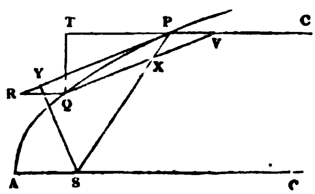
Scholium to Prop. X.

1. It was proved in the proposition, that, when a body moves in an ellipse round a center of force in the center, the force varies as the distance. The same is also true, when a body moves in an hyperbola, the construction and proof being exactly the same as for the ellipse.

2. If the orbit be a parabola, the center of force is removed to an infinite distance, and the force acts in lines parallel to the axis; in this case, since the difference of any two distances vanishes compared with the distances themselves, the force is invariable.

Or the following proof may be applied in the case of the parabola.

Let PQ be an arc of the parabola, A the vertex, S the focus; PC parallel to the axis, and therefore in the direction of the force; QR a subtense parallel to PC, and QV parallel to the tangent PR; QT, SY perpendicular to CP, PR.



Since $4SP \cdot PV = QV^2$, $\dfrac{QR}{QV^2} = \dfrac{PV}{QV^2} = \dfrac{1}{4SP}$,

and by similar triangles, QTV, SPY,

$$\frac{QV^2}{QT^2} = \frac{SP^2}{SY^2} = \frac{SP}{SA};$$

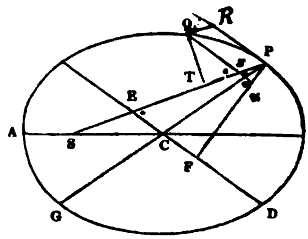
$$\therefore \text{ limit } \frac{QR}{QT^2} = \frac{1}{4SA};$$

$$\therefore F = \frac{2h^2}{CP^2} \cdot \frac{1}{4SA}.$$

SECTION III.

ON THE MOTION OF BODIES IN CONIC SECTIONS, ABOUT A CENTER OF FORCE IN ONE OF THE FOCI.

PROP. XI. *A body revolves in an ellipse, to find the law of force tending to one of the foci.*



Let the focus S be the center of force, PQ an arc, QR a subtense parallel to SP; C the center of the ellipse, join PC and produce it to meet the ellipse in G; draw Qxv parallel to the tangent PR, cutting SP, CP in x, v; and QT, PF respectively perpendicular to SP, and the semi-conjugate diameter CD: and let E be the point, in which SP cuts CD, then $PE = AC$ the $\frac{1}{2}$ axis major.

By similar triangles, QxT, PEF,

$$\frac{Qx^2}{QT^2} = \frac{PE^2}{PF^2} = \frac{AC^2}{PF^2},$$

and by a property of the ellipse,

$$\frac{Pv}{Qv^2} = \frac{CP^2}{vG \cdot CD^2};$$

also by similar triangles, PxV, PEC,

$$\frac{Px}{Pv} = \frac{PE}{PC} = \frac{AC}{PC}.$$

Now $Px = QR$, Qx ultimately $= Qv$, and vG ultimately $= 2CP$; hence multiplying the above quantities together, and taking the limits of the products,

$$\text{limit } \frac{QR}{QT^2} = \frac{AC^3 \cdot CP}{2CP \cdot CD^2 \cdot PF^2} = \frac{AC^3}{2AC^2 \cdot BC^2},$$

$$= \frac{AC}{2BC^2} = \frac{1}{L};$$

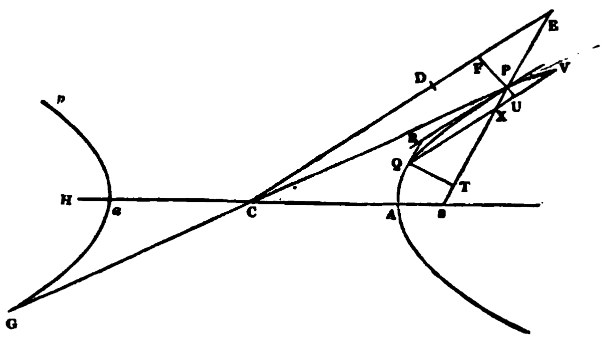
$$\therefore F = \frac{2h^2}{SP^2} \text{ limit } \frac{QR}{QT^2}$$

$$= \frac{2h^2}{L} \frac{1}{SP^2} \text{ or } = \frac{\mu}{SP^2}$$

$$\propto \frac{1}{SP^2}.$$

PROP. XII. *A body moves in an hyperbola, to find the law of force tending to one of the foci.*

Let the center of force be in the focus S, and let the body move in the branch PA, which is nearer to S than

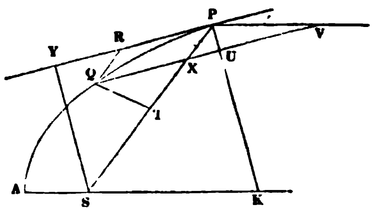


the other branch of the hyperbola. Then the same construction being made as in the ellipse, it may be shewn in precisely the same manner that the force

$$= \frac{2h^2}{L} \cdot \frac{1}{SP^2} \text{ or } = \frac{\mu}{SP^2} \text{ and } \therefore \propto \frac{1}{SP^2}.$$

Cor. In the same manner it may be shewn, that if the body describes the opposite branch pa by a ***repulsive*** force proceeding from S, the force will vary as $\frac{1}{Sp^2}$.

Prop. XIII. ***A body moves in a parabola, to find the law of force tending to the focus.***



Let A and S be the vertex and focus of the parabola, PQ an arc; QR a subtense parallel to SP, QXV parallel to the tangent PR, cutting SP in X, and the diameter through P in V; QT, SY perpendicular to SP, PR; L = latus rectum. Then by a property of the parabola $PV = PX$ and $\therefore = QR$; also $4SP.PV = QV^2$.

Hence

$$\frac{QR}{QV^2} = \frac{PV}{QV^2} = \frac{1}{4SP},$$

and by similar triangles,

$$\frac{QX^2}{QT^2} = \frac{SP^2}{SY^2} = \frac{SP}{SA}.$$

Now QX ultimately $= QV$; hence multiplying these quantities together, and taking the limits,

$$\text{limit } \frac{QR}{QT^2} = \frac{1}{4SA} = \frac{1}{L},$$

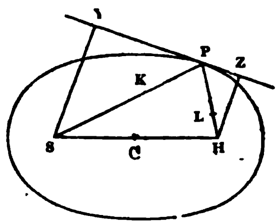
$$\therefore F = \frac{2h^2}{SP^2} \text{ limit } \frac{QR}{QT^2},$$

$$= \frac{2h^2}{L} \cdot \frac{1}{SP^2} \text{ or } = \frac{\mu}{SP^2},$$

$$\propto \frac{1}{SP^2}.$$

Cor. *If a body be projected at a given distance from a center of force, which $\propto (dist.)^{-2}$, and in a direction making a finite angle with the distance, it will describe a conic section.*

Let S be the center of force, P the point and PY the direction of projection, $F=$ the force at P, then if s be the space due to the velocity of projection, $s=\dfrac{(\text{velocity})^2}{2F}$, and is therefore known.



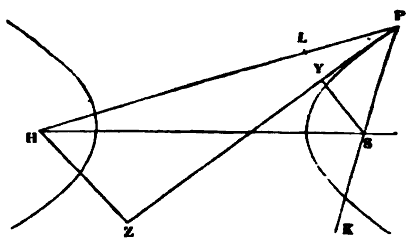
1. Let s be less than SP. In PS take $PK=s$, and draw PH, making with YP produced the $\angle HPZ = \angle SPY$; in PH take $PL = SK$, and let a circle described through the points S, K, L cut PL in H, then $PH \cdot PL = PS \cdot PK$. With foci S and H and axis major $= SP + HP$, describe an ellipse, and suppose a body revolving in this ellipse and acted on by the same force in S, to come to P; then space due to velocity at $P = \frac{1}{4}$ chord of curvature at P through S,

$$= \frac{(\frac{1}{2}\text{ conjugate diameter})^2}{\text{axis major}} = \frac{SP \cdot HP}{SP + HP},$$

$$= \frac{SP}{1+\dfrac{SP}{HP}} = \frac{SP}{1+\dfrac{PL}{PK}} = PK.$$

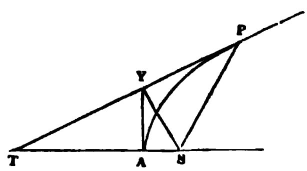
Hence the velocity is the same in both cases; also the revolving body is moving in direction PY, since ZPy, making equal angles with SP, HP, is a tangent at P; and the force and the law of force are the same for both bodies; they will therefore describe the same curve, that is, the projected body will describe an ellipse.

2. Let s be greater than SP. In PS produced take $PK = s$; draw PH on the other side of PY, making the $\angle\, YPH = \angle\, YPS$, take $PL = SK$, and let a circle described through the points S, K, L cut PL produced in H: then if with foci S and H and axis major $= HP \sim SP$, an hyperbola be described, it may be shewn, as in the preceding case, that the body will move in the hyperbola thus constructed.



3. Let $s = SP$. Here $SK = 0$, and $\therefore PH = \dfrac{PK.PS}{SK} = \infty$.

Let the circle described with center S and radius SP cut PY in T, join TS; draw SY, YA perpendicular to PT, TS respectively, and with focus S and vertex A describe a parabola; it will pass through the point P; for a parabola, whose focus is S, and which passing through P has PY for a tangent, will have its axis coincident with ST, and its latus rectum will $= 4\dfrac{SY^2}{SP}$, which $= 4SA$; hence, conversely, the parabola above described will pass through P, and it may be shewn as in the former cases, that the body will move in this parabola.



Prop. XIV. *If any number of bodies revolve about one common center of force, which varies as* $(dist.)^{-2}$, *and is the same at equal distances in all the orbits described, the latera recta of the orbits will be as the squares of the areas described in equal times.*

Let $\dfrac{\mu}{SP^2}$ be the force in any orbit at the distance SP, then since the forces at equal distances are equal, μ is the same for all the orbits:

Also by Props. XI, XII, XIII, $\mu = \dfrac{2h^2}{L}$,

$$\therefore L \propto h^2 \propto \left(\frac{\text{area described in a given time}}{\text{time}}\right)^2$$

$$\propto (\text{area})^2 \text{ described in a given time.}$$

PROP. XV. *A body revolves in an ellipse round a center of force in the focus, to find the periodic time.*

Let AC, BC be the semi-axes major and minor, P the periodic time.

$$\text{Then } \frac{P''}{1''} = \frac{\text{area of the ellipse}}{\text{area described in } 1''},$$

periodic time² ∝ mean dist³

$$\therefore P = \frac{\pi AC \cdot BC}{\frac{1}{2}h},$$

or $P \propto (ac)^{\frac{3}{2}}$

$$\text{and since } \frac{2h^2}{L} = \mu,\ h = \sqrt{\frac{\mu L}{2}} = \sqrt{\frac{\mu \cdot BC^2}{AC}} = BC\sqrt{\frac{\mu}{AC}},$$

$$\therefore P = \frac{2\pi AC^{\frac{3}{2}}}{\mu^{\frac{1}{2}}}.$$

COR. Hence, the squares of the periodic times in all ellipses, described round the same center of force in the focus, are as the cubes of the major axes.

PROP. XVI. *To find the velocity at any point of a conic section, described about a center of force in the focus.*

Let V be the velocity at the point P,

$$V^2 = \tfrac{1}{2}F \cdot PV = \frac{\mu}{SP^2} \cdot \frac{PV}{2}.$$

Now in the ellipse and hyperbola,

$$\frac{PV}{2} = \frac{CD^2}{AC} = \frac{SP \cdot HP}{AC} = SP \cdot \left(2 \mp \frac{SP}{AC}\right),$$

and in the parabola,

$$\frac{PV}{2} = 2SP.$$

Hence in the ellipse $V = \sqrt{\frac{\mu}{SP}\left(2 - \frac{SP}{AC}\right)}$,

in hyperbola $V = \sqrt{\frac{\mu}{SP}\left(2 + \frac{SP}{AC}\right)}$,

in parabola $V = \sqrt{\frac{2\mu}{SP}}$.

Cor. To compare the velocity at P with that of a body moving in a circle, radius $= SP$, and described round the same center of force.

Let $U =$ velocity in the circle,

then (Prop. VI. Cor. 5),

$$U = \sqrt{F.R} = \sqrt{\frac{\mu}{SP^2} SP.} = \sqrt{\frac{\mu}{SP}},$$

$\therefore$ in ellipse $\frac{V}{U} = \sqrt{2 - \frac{SP}{AC}}$ which is less than $\sqrt{2}$,

in hyperbola $\frac{V}{U} = \sqrt{2 + \frac{SP}{AC}}$........greater.........

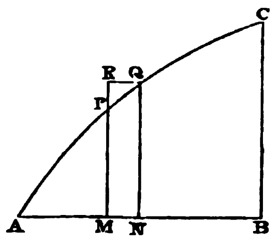
in parabola $\frac{V}{U} = \sqrt{2}$.

APPENDIX.

NOTE TO LEMMA II.

1. To find the area of a plane curve.

Let the area ABC be bounded by the curve AC and the straight lines AB, BC. Let AB be divided into n equal parts, and let MN be the r^{th} part from A; draw MP, NQ parallel to BC, and complete the parallelogram $MNQR$.



Let $AB = h$, then $MN = \frac{h}{n}$,

$NQ = y_r$,

$\angle ABC = i$,

$$\text{area of parallelogram } RN = \frac{h}{n} y_r \sin i.$$

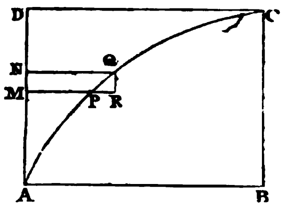
Therefore giving to r the values 1, 2, 3...n, the sum of the parallelograms described on all the parts

$$= \frac{n}{h} \sin i \,(y_1 + y_2 + y_3 + \ldots + y_n) = h \sin i \,.\, \Sigma \frac{y_r}{n}.$$

Therefore area of curvilinear figure $= h \sin i \,.\, \text{limit}\, \Sigma \frac{y_r}{n}$,

when n is infinite.

Ex. 1. To find the area of a portion of a parabola cut off by a diameter and one of its ordinates.



Let ABC be the parabolic area cut off by the diameter AB and a semi-ordinate BC. Complete the parallelogram $ABCD$; then AD is a tangent at A.

Let $AD = h$, $AB = k$, and let AN be the abscissa, and NQ, parallel to AB, the ordinate to the point Q; then by a property of the parabola,

$$\frac{QN}{AN^2} = \frac{AB}{AD^2}; \ \therefore QN \text{ or } y_r = \frac{k}{h^2} \cdot \left(\frac{rh}{n}\right)^2 = k\frac{r^2}{n^2},$$

$$\therefore \text{ area } ADC = h \sin i \,.\, \text{limit} \,.\, \Sigma \frac{y_r}{n} = kh \sin i \text{ limit } \Sigma \frac{r^2}{n^3},$$

$$= hk \sin i \,.\, \text{limit} \,.\, \frac{1}{n^3} \cdot (1^2 + 2^2 + 3^2 + \ldots + n^2),$$

$$= hk \sin i \,.\, \text{limit } \frac{1}{n^3}\left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right)$$

$$= \tfrac{1}{3} hk \sin i$$

$$= \tfrac{1}{3} \text{ parallelogram } ABCD,$$

and $\therefore$ parabolic area $ABC = \frac{2}{3}$ circumscribing parallelogram.

2. The volume of a solid of revolution may be deduced in a similar manner.

Let ABC (Fig. Art. 1.) be a plane curvilinear area by the revolution of which round AB the solid is generated, and let CB be perpendicular to AB. Then if AB ($= h$) be divided into n equal parts, and the rectangular parallelogram RN be described on MN the r^{th} part, the cylinder generated by the revolution of RN round MN

$$= \frac{h}{n} \pi \,.\, QN^2 = \frac{h}{n} \pi \,.\, y_r^2,$$

and the volume of the solid

$$= \text{limit . sum of all such cylinders}$$

$$= \pi h \,.\, \text{limit}\, \Sigma \,.\, \frac{y_r^2}{n}, \text{ when } n \text{ is infinite.}$$

Ex. 2. To find the volume of a sphere.

Let ABC be a quadrant of the generating circle radius $= h$;

$$\text{then } y^2 = 2hx - x^2,$$

$$y_r^2 = 2h\,\frac{rh}{n} - \left(\frac{rh}{n}\right)^2 = h^2\left(\frac{2r}{n} - \frac{r^2}{n^2}\right),$$

and therefore volume of hemisphere

$$= \pi\, h^3 \text{ limit}\left\{\frac{2}{n^2}(1 + 2 + \dots n) - \frac{1}{n^3}(1^2 + 2^2 + \dots + n^2)\right\}$$

$$= \pi\, h^3 \text{ limit}\left\{\frac{2}{n^2}\left(\frac{n^2}{2} + \frac{n}{2}\right) - \frac{1}{n^3}\left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right)\right\}$$

$$= \pi h^3\left(1 - \frac{1}{3}\right) = \frac{2}{3}\pi h^3,$$

and therefore volume of sphere

$$= \frac{4}{3}\pi h^3 = \frac{2}{3} \,.\, 2h \,.\, \pi h^2 = \frac{2}{3} \text{ circumscribing cylinder.}$$

Ex. 3. Similarly the volume of a cone and of a paraboloid may be shewn to be $\frac{1}{3}$ and $\frac{1}{2}$ of the circumscribing cylinder respectively.

3. To find the volume of a pyramid.

Let A be the area of the base of the pyramid, and let the perpendicular from the vertex upon the base $= h$. Divide h into n equal parts, and through the r^{th} point of division from the vertex draw a plane parallel to the base. Then the area of the section of the pyramid thus made

$$= A\frac{\left(\frac{rh}{n}\right)^2}{h^2} = A\frac{r^2}{n^2};$$

on this area as a base describe a right prism, whose altitude $= \frac{h}{n}$; then volume of prism

$$= A\frac{r^2}{n^2}\cdot\frac{h}{n} = Ah\frac{r^2}{n^3};$$

and therefore volume of pyramid = limit of sum of all such prisms

$$= Ah \text{ limit } \Sigma\frac{r^2}{n^3} = Ah \text{ limit } \frac{1}{n^3}(1^2 + 2^2 + \ldots\ldots + n^2)$$

$$= Ah \text{ limit } \frac{1}{n^3}\left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right)$$

$$= \tfrac{1}{3}Ah = \tfrac{1}{3}\,.\,\text{base} \times \text{altitude}.$$

Note to Prop. III. on Curvature.

4. To find the chords of curvature through the center and focus, and the diameter of curvature, at any point of an ellipse and hyperbola. (Vide Figs. Prop. XI. and XII.)

Let Qv, a semi-ordinate to the diameter PCG, cut SP in x, CP in v, and PF, which is perpendicular to the semi-conjugate CD, in u.

Chord of curvature through C

$$= \text{limit} \frac{PQ^2}{\text{subtense parallel to } CP}$$

$$= \text{limit} \frac{PQ^2}{Pv} = \text{limit} \frac{Qv^2}{Pv}$$

$$= \text{limit} \frac{CD^2}{CP^2} \cdot vG, \quad \text{since } \frac{Qv^2}{Pv \cdot vG} = \frac{CD^2}{CP^2}$$

$$= \frac{2CD^2}{CP}, \text{ since } vG \text{ ultimately} = 2CP.$$

Chord of curvature through S

$$= \text{limit} \frac{PQ^2}{\text{subtense parallel to } SP} = \text{limit} \frac{Qv^2}{Px}$$

$$= \text{limit} \frac{Qv^2}{Pv} \cdot \frac{Pv}{Px} = \text{limit} \frac{CD^2}{CP^2} \cdot vG \cdot \frac{PC}{PE}$$

$$= \frac{2CD^2}{AC}, \text{ since } PE = AC.$$

Diameter of curvature

$$= \text{limit} \frac{PQ^2}{\text{subtense perpendicular to tangent}}$$

$$= \text{limit} \frac{Qv^2}{Pu} = \text{limit} \frac{Qv^2}{Pv} \cdot \frac{Pv}{Pu} = \text{limit} \frac{CD^2}{CP^2} \cdot vG \cdot \frac{PC}{PF}$$

$$= \frac{2CD^2}{PF}.$$

COR. Let PF cut the axis major in K, then $PF . PK = BC^2$;

also $CD . PF = AC . BC$,

$$\therefore \text{ diameter of curvature } = \frac{2AC^2 . BC^2}{PF^3} = \frac{2AC^2 . BC^2 . PK^3}{BC^6}$$

$$= \frac{8PK^3}{L^2}.$$

5. To find the chord of curvature through the focus, and the diameter of curvature at any point of a parabola. (Vide Fig. Prop. XIII.)

Let QV, a semi-ordinate to the diameter PV, cut SP in X, and the normal PK in U, draw SY perpendicular to the tangent at P; then $PX = PV$, hence

Chord of curvature through S

$$= \text{limit } \frac{PQ^2}{\text{subtense parallel to } SP}$$

$$= \text{limit } \frac{PQ^2}{PX} = \text{limit } \frac{QV^2}{PV}$$

$$= 4SP, \text{ since } QV^2 = 4SP . PV.$$

Diameter of curvature

$$= \text{limit } \frac{PQ^2}{\text{subtense perpendicular to tangent}} = \text{limit } \frac{QV^2}{PU}$$

$$= \frac{4SP . PV}{PU} = 4SP . \frac{SP}{SY}, \text{ by sim}^r. \text{ triangles } PVU,\ SPY,$$

$$= \frac{4SP^2}{SY}, \text{ or } = 4\sqrt{\frac{SP^3}{SA}}.$$

PK, the normal.

Cor. Let PY meet the axis in T, then

$$ST = SP = SK, \therefore SY = \tfrac{1}{2} PK;$$

hence diameter of curvature

$$= 4\,\frac{SP^2}{SY} = 4\,\frac{SY^4}{SA^2 . SY} = \frac{4}{SA^2}\,SY^3 = \frac{1}{2SA^2} . PK^3$$

$$= \frac{8\,PK^3}{L^2}.$$

NOTE TO PROP. VI. COR. 3.

6. If $SP = r$, and $SY = p$, $PV = \dfrac{2p}{d_r p}$.

$$\therefore\ F = \frac{2h^2}{SY^2 . PV} = \frac{2h^2}{p^2 . \dfrac{2p}{d_r p}}$$

$$= \frac{h^2}{p^3} . d_r p.$$

Again, if $r = \dfrac{1}{u}$, and $ASP = \theta$, AS being a fixed straight line drawn through S,

$$\frac{1}{p^2} = (d_\theta u)^2 + u^2;$$

$$\therefore\ \frac{1}{p^3} d_r p = -\frac{1}{p^3} . d_u p . u^2 = -\frac{1}{p^3}\,\frac{d_\theta p}{d_\theta u} . u^2$$

$$= (d_\theta u . d_\theta^2 u + u\,d_\theta u)\,\frac{u^2}{d_\theta u}$$

$$= u^2 (d_\theta^2 u + u);$$

$$\therefore\ F = h^2 u^2 (d_\theta^2 u + u).$$

Ex. 1. To find the law of force, by which a body may describe the curve, whose equation is $p = \dfrac{ar}{\sqrt{b^2 + r^2}}$, round a center of force in the pole.

$$\frac{1}{p^2} = \frac{b^2}{a^2 r^2} + \frac{1}{a^2},$$

$$\text{and } \therefore \frac{1}{p^3} d_r p = \frac{b^2}{a^2 r^3};$$

$$\therefore F = \frac{h^2}{p^3} d_r p = \frac{h^2 b^2}{a^2 r^3} \propto \frac{1}{r^3}.$$

Ex. 2. To find the law of force by which a body may describe a conic section, round a center of force in the focus.

Let P be any point in the curve, S the focus, A the extremity of the axis major;

$$SP = r = \frac{1}{u}, \quad ASP = \theta,$$

$$\text{then } r = \frac{m}{1 + e\cos\theta};$$

$$\therefore u = \frac{1}{m}(1 + e\cos\theta),$$

$$d_\theta u = -\frac{1}{m} e \sin\theta,$$

$$d_\theta^2 u = -\frac{1}{m} e\cos\theta;$$

$$\therefore F = h^2 u^2 (d_\theta^2 u + u)$$

$$= h^2 u^2 \frac{1}{m}$$

$$= \frac{h^2}{m r^2} \propto \frac{1}{r^2}.$$

NOTE TO PROP. X. COR. 3.

7. To find the magnitude and position of the axes of the orbit described.

Let $CP = r$, $CPy = \alpha$, $s \left(= \frac{V^2}{2F}\right)$ = space due to velocity of projection, a and b the semi-axes of the orbit described.

$$CD = \sqrt{\tfrac{1}{2} CP \cdot PV} = \sqrt{2rs},$$

$$\left.\begin{aligned} a^2 + b^2 (= CP^2 + CD^2) &= r^2 + 2rs \\ ab \ \ (= CD \cdot PF) &= \sqrt{2rs} \cdot r \sin \alpha \end{aligned}\right\},$$

from which two equations a and b, and therefore e, the eccentricity, may be determined.

Also, if θ be the inclination of axis major to CP,

$$r^2 = \frac{b^2}{1 - e^2 \cos^2 \theta},$$

$$\therefore \cos^2 \theta = \frac{1}{e^2}\left(1 - \frac{b^2}{r^2}\right),$$

from which θ may be found.

NOTE TO PROP. XIII. COR.

8. To find the magnitude and position of the axis of the orbit described.

Let $SP = r$, $\angle SPY = \alpha$, draw SY, HZ perpendicular to YPZ; and let a, b, L be the semiaxes and latus rectum of the orbit;

$$\text{then } PL \cdot PH = PK \cdot PS,$$

$$\text{or } (r \sim s) \cdot (2a \mp r) = s \cdot r,$$

$$\therefore 2a(r \sim s) - r^2 = 0,$$

$$\therefore\ a = \frac{r^2}{2(r \sim s)},$$

hence the magnitude of the axis major is independent of the direction of projection.

$$\text{Again, } b = \sqrt{SY \,.\, HZ} = \sqrt{SP \sin \alpha \,.\, HP \sin \alpha}$$

$$= \frac{r \,.\, s^{\frac{1}{2}} \,.\, \sin \alpha}{(r \sim s)^{\frac{1}{2}}};$$

$$\therefore\ L = \frac{2b^2}{a}$$

$$= 4s \,.\, \sin^2 \alpha.$$

Again, $$YR = SH \,.\, \sin ASY,$$

$$\text{or } (SP + HP) \cos \alpha = 2e \,.\, AC \sin ASY,$$

$$\therefore\ \sin ASY = \frac{1}{e} \cos \alpha,$$

which equation, since $e = \sqrt{1 \mp \frac{b^2}{a^2}}$ is known, determines the position of the axis major.

ON ANGULAR VELOCITY.

2. When a body P moves in an orbit, its angular velocity round any point S, (fig. page 28.) is measured by the angle uniformly described by SP round S in 1″, in the same manner as linear velocity is measured by the line uniformly described in 1″. If the angular motion of SP be not uniform, the angular velocity at any point is measured by the angle, which would be described in 1″, if the angular motion of SP were to continue uniform for that time. Hence if the angular motion be not uniform, and PSQ be the angle described in T'' after leaving P, the angular velocity

$$= \text{limit}\ \frac{\text{angle } PSQ}{T},$$

for this is the angle which would be described in 1″, if the angular motion at P were to continue uniform for that time.

Prop. *If a body be moving in any orbit round a center of force S, the angular velocity at any point P*

$$= \frac{h}{SP^2}.$$

Let PSQ be the angle described in T''; with center S and radius SQ, describe a circular arc cutting SP in T, and draw SY perpendicular to the tangent at P; then the triangle PTQ may be considered as ultimately rectilinear, and similar to SYP, hence

$$\angle^r \text{ vel. at } P = \text{limit} \frac{\angle PSQ}{T} = \text{limit} \frac{QT}{SQ \,.\, T}$$

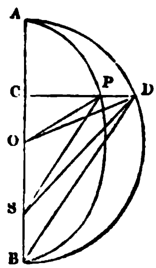
$$= \text{limit} \frac{PQ \,.\, SY}{SP^2 \,.\, T}, \text{ since limit } \frac{QT}{PQ} = \frac{SY}{SP}$$

$$= \frac{SY \,.\, \text{vel. at } P}{SP^2}, \text{ since limit } \frac{PQ}{T} = \text{vel. at } P$$

$$= \frac{h}{SP^2}, \text{ (Prop. I. Cor. 3).}$$

---

10. *Force varying as* $(distance)^{-2}$. *To find the time of motion and the velocity acquired by a body falling through a given space from rest.* (Props. XXXIII. and XXXVI.)



Let S be the center of force, A the point from which the body begins to fall;

$$\frac{\mu}{SP^2} = \text{force at distance } SP.$$

Let APB be a semi-ellipse, focus S and axis major ASB; ADB a semi-circle, whose diameter is ASB; and suppose a body revolving in the

ellipse round the focus S to come to P; bisect AB in O, draw DPC perpendicular to AB, and join OP, OD.

Then the time through $AP \propto$ area $ASP \propto$ area ASD, and this being true for all values of the axis minor will be true when it is diminished without limit, in which case the ellipse coincides with the axis major and the point P with C, or the body is moving in the straight line AC; the point B also coincides with S, since $AS \cdot SB = (\frac{1}{2}$ axis minor$)^2$; and since space due to velocity at $A = \frac{1}{4}$ chord of curvature at A through $S = \frac{1}{4}$ latus rectum $= \dfrac{(\text{axis minor})^2}{4AB} = 0$, the body begins to move from rest at A.

Hence time from rest through $AC \propto$ area ABD,

$$\therefore \frac{\text{time through } AC}{\text{time through } AB (= \frac{1}{2} \text{periodic time in ellipse})} = \frac{\text{area } ABD}{\text{semi-circle } ABD};$$

$$\therefore \text{ time through } AC = \frac{\pi \cdot AO^{\frac{3}{2}}}{\sqrt{\mu}} \cdot \frac{\frac{1}{2} AO \cdot (AD + CD)}{\frac{1}{2} \pi \cdot AO^2}$$

$$= \sqrt{\frac{AS}{2\mu}} \cdot (AD + CD).$$

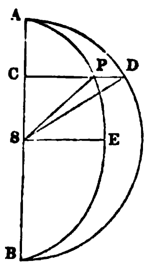
Again, velocity at $P = \sqrt{\dfrac{\mu}{AO} \cdot \dfrac{HP}{SP}}$ (Prop. XVI.), and when the ellipse coincides with the axis major,

$$\text{velocity at } C = \sqrt{\frac{2\mu}{AS} \cdot \frac{AB - BC}{BC}} = \sqrt{\frac{2\mu}{AS} \cdot \frac{AC}{SC}}.$$

Cor. Time through $AS = \sqrt{\dfrac{AS}{2\mu}}\, \pi \dfrac{AS}{2} = \dfrac{\pi \left(\dfrac{AS}{2}\right)^{\frac{3}{2}}}{\sqrt{\mu}}$

$= \frac{1}{2}$ per. time in an ellipse, of which AS is the axis major.

11. *Force varies as distance. To find the time of motion and the velocity acquired by a body in falling through a given space from rest.* (Prop. xxxviii.)



Let S be the center of force, A the place from which the body begins to fall: on $AB = 2AS$ describe a semi-ellipse APB, and a semi-circle ADB, and let a body moving in the ellipse come to P. Draw DPC perpendicular to AB, and join SP, SD.

Then time through $AP \propto$ area $ASP \propto$ area ASD, and this being true, whatever be the axis minor of the ellipse, will be true when it is diminished without limit, in which case the body will be at C, having fallen from rest at A,

$\therefore$ time through $AC \propto$ area ASD;

$$\therefore \frac{\text{time through } AC}{\text{time through } AS (= \frac{1}{4} \text{periodic time in a circle})}$$

$$= \frac{\text{sector } ASD}{\frac{1}{4} \text{area of a circle}};$$

$$\therefore \text{time through } AC = \frac{\pi}{2\sqrt{\mu}} \cdot \frac{\frac{1}{2}AS \cdot AD}{\frac{1}{4}\pi AS^2}$$

$$= \frac{AD}{AS\sqrt{\mu}}.$$

Again, let SE be the semi-axis minor,

then vel. at $P =$ semi-conjugate at $P \cdot \sqrt{\mu}$ (Prop. x. Cor. 1.)

$$= \sqrt{AS^2 + SE^2 - SP^2} \cdot \sqrt{\mu},$$

$$\therefore \text{vel. at } C = \sqrt{AS^2 - SC^2} \cdot \sqrt{\mu}$$

$$= CD\sqrt{\mu}.$$

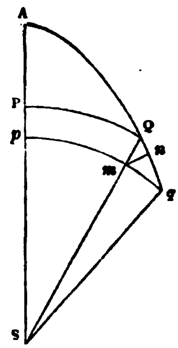
Cor. Time to center of force $= \dfrac{\frac{1}{2}\pi AS}{AS\sqrt{\mu}} = \frac{1}{4}\dfrac{2\pi}{\sqrt{\mu}}$

$= \frac{1}{4}$ per. time in an ellipse, force in center.

Hence the times through all distances to the center of force are equal.

Vel. acquired in falling through $AS = AS\sqrt{\mu}$.

12. *If the velocities of two bodies, one of which is falling directly towards a center of force, and the other describing a curve about that center, be equal at any equal distances, they will always be equal at equal distances.* (Prop. xl.)



Let S be the center of force, and let one of the bodies be moving in the straight line APS and the other in the curve AQq; with radii SQ, Sq describe the circular arcs QP, qp: let SQ cut pq in m, and draw mn perpendicular to Qq; and suppose the velocities of the bodies at P and Q to be equal.

Since the centripetal forces at P and Q are equal, Pp, Qm may be taken to represent them: Pp is wholly effective in accelerating P, but the effective part of Qm is Qn, nm being wholly employed in retaining the body in the curve. Also since the velocities at P and Q are equal, the times of describing Pp and Qq, when the spaces are diminished indefinitely, are proportional to Pp and Qq; hence

$$\text{force at } P : \text{force at } Q = Pp : Qn$$

and time through Pp : time through $Qq = Pp : Qq$;

$\therefore$ velocity added in describing Pp : velocity added in describing Qq

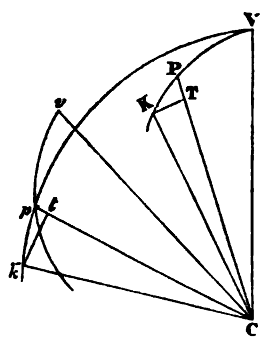
$$= Pp^2 : Qn \,.\, Qq = Qm^2 : Qn \,.\, Qq$$
$$= 1 : 1,$$

and the same may be shewn at all corresponding points equally distant from S, therefore, *If the velocities, &c.*

SECTION IX.

ON THE POSITION OF THE APSIDES IN ORBITS VERY NEARLY CIRCULAR.

PROP. XLIII. *The orbit in which a body moves revolves round the center of force with an angular velocity, which always bears a fixed ratio to that of the body; to shew that the body may be made to move in the revolving orbit in the same manner as in the orbit at rest by the action of a force tending to the same center.*



Let C be the center of force, and when the body in the fixed orbit VCP has described the arc VP, let vCp be the position of the revolving orbit, and p that of the body moving in it; then $\angle\, vCp = \angle\, VCP$. Also let the angular velocity of the orbit be to that of P as $G - F : F$.

The angles VCv, VCP begin together at V, and their contemporary increments are as the angular velocities of Cv and CP, that is, as $G - F : F$, therefore the angles themselves are in that ratio, or

$$VCv : VCP \text{ or } vCp = G - F : F;$$

$\therefore$ componendo $VCp : VCP = G : F$;

hence, if the angle VCp be always taken $= \dfrac{G}{F} \times$ angle VCP,

and $Cp = CP$, Vp the locus of p will be the curve traced out in fixed space by a body p moving in the revolving orbit in the same manner as P in the fixed orbit.

Also the body may describe the orbit Vp by the action of a force placed in C.

For let PCK, pCk be the areas described by CP, Cp in the same small increment of time; draw KT, kt perpendicular to CP, Cp; then the contemporary increments of the areas, described by p and P, are ultimately as

$$Cp\,.\,kt : CP\,.\,KT = Cp^2\,.\sin pCk : CP^2\,.\sin PCK$$
$$= \angle\, pCk : \angle\, PCK = \angle^r \text{ vel. of } Cp : \angle^r \text{ vel. of } CP = G : F,$$

and the whole areas begin together at V, therefore they are themselves in the same ratio; hence area $VCp \propto$ area VCP $\propto$ the time (Prop. 1); and therefore (Prop. 2) a body may be made to move in the orbit Vp by a proper centripetal force placed in C.

Def. An apse or apside is a point in an orbit at which the direction of the body's motion is perpendicular to the distance; and the angle between two consecutive apsidal distances is called the apsidal angle.

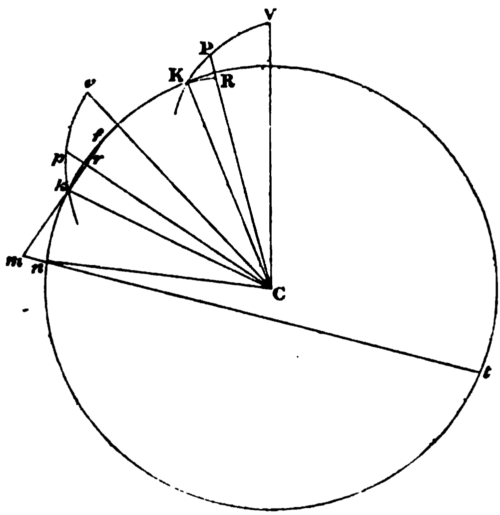
Cor. *If a be the apsidal angle in the orbit* VP, *the corresponding apsidal angle in the orbit* $\mathrm{Vp} = \dfrac{\mathrm{G}}{\mathrm{F}}\,a$.

For the motion of p is compounded of two motions, one arising from the angular motion of the orbit, and therefore perpendicular to the distance, and the other the same as the motion of P in the fixed orbit; hence when the latter body is at an apse, the whole motion of p will be perpendicular to the distance, or p will be at an apse; also the angles described in the same time in the orbits Vp and VP are always as $G : F$,

$$\therefore \text{ apsidal angle in orbit } Vp = \frac{G}{F}\,a.$$

Prop. XLIV. *To find the difference of the forces, by which the bodies are retained in the fixed and revolving orbits.*

Let P and p be contemporary positions of the two bodies, PK a small arc of the fixed orbit described in t''; take $pk = Pk$, and with radius CK or Ck describe the circle Kk;



draw KR, kr perpendicular to CP, Cp and in rk, produced if necessary, take $rm = \frac{G}{F}.kr$. Let the velocities of P and p be each resolved into two, one central or in the direction of the distance, and the other transverse or perpendicular to it; then since PK is very small, PR and RK may be taken to represent P's central and transverse velocities respectively; and since the angular motion of the orbit affects only the transverse motion of p, $pr = PR$ will represent p's central motion: also transverse vel. = angular vel. × dist.;

$\therefore$ transv. vel. of p : transv. vel. of P = $\angle^r$ vel. of p : $\angle^r$ vel. of P

$= G : F$;

$$\therefore \text{ transverse vel. of } p = \frac{G}{F} . KR = rm.$$

Hence, in consequence of the two motions pr, rm, p will be at m, when P is at K. But if we take

$$\angle VCn = \frac{G}{F} \angle VCK, \quad \text{and} \quad Cn = CK,$$

p must be at n, when P is at K, in order that it may move in the manner required; join mn; then an additional force must have acted on p, sufficient to draw it through mn in t'', and therefore the difference of the forces on P and p

$$= 2 \lim. \frac{mn}{t^2} \quad \text{(Lem. x. Cor. 2).}$$

Let mn, mr produced cut the circle again in t and f,

then $mn = \dfrac{mk . mf}{mt}$. Now $mr = \dfrac{G}{F} . kr$, $\therefore mk = \dfrac{G - F}{F} kr$,

and $mf = \dfrac{G + F}{F} . kr$; $\therefore mk . mf = \dfrac{G^2 - F^2}{F^2} . kr^2$.

Let $h = 2$ area described by P in $1''$,

$$\therefore h = 2 \lim. \frac{\text{area } PCK}{t} = \lim. \frac{CP . KR}{t}; \quad \therefore \lim. \frac{KR}{t} = \frac{h}{CP};$$

also mt ultimately passes through C and equals $2CP$;

$$\therefore 2 . \lim. \frac{mn}{t^2} = 2 . \lim. \frac{G^2 - F^2}{F^2} . \frac{kr^2}{t^2 . 2CP};$$

$$\therefore \text{force on } p - \text{force on } P = \frac{G^2 - F^2}{F} . \frac{h^2}{CP^3}, \text{ and } \therefore \propto \frac{1}{CP^3}.$$

PROP. XLV. *The law of force in an orbit nearly circular being given, to find an approximate value of the apsidal angle.*

Let $\frac{1}{r^3}.fr$ be the force at any distance r, a the greatest value of r, and $a-x$ any other value; then

$$\frac{1}{r^3}fr = \frac{1}{r^3}f(a-x),$$

which being expanded in a series ascending by powers of x

$$= \frac{1}{r^3}(fa - f'a \,.\, x + \text{\&c.}) = r^3(fa - f'a \,.\, x) \text{ very nearly,}$$

since x is very small.

Let VP (Fig. Prop. XLIII.) be an ellipse of small eccentricity, C the focus, CV the greatest distance $= a$, L the latus rectum, and let $\frac{F^2}{a^2} =$ force at V; then (Prop. XI.) if $h = 2$ area described in $1''$ by a body revolving in the ellipse round a center of force in the focus,

$$F^2 = \frac{2h^2}{L} = \frac{h^2}{a}, \text{ since } L = 2a \text{ nearly; hence,}$$

$$\text{force on } p = \frac{F^2}{Cp^2} + \frac{G^2 - F^2}{F^2} \cdot \frac{h^2}{Cp^3} \text{ (Prop. XLIV.)}$$

$$= \frac{1}{Cp^3}\{F^2\, Cp + (G^2 - F^2)\, a\}$$

$$= \frac{1}{r^3}\{F^2(a-x) + (G^2 - F^2)\, a\} \text{ since } Cp = r \text{ or } a - x,$$

$$= \frac{1}{r^3}(G^2 a - F^2 x).$$

Now the values of G and F being indeterminate, this expression may be made equal to the above value of the force in the orbit, of which the apsidal angle is required, that is,

$$G^2 a - F^2 x = fa - f'ax,$$

from which equation, since it must hold true for the different values of x, we obtain

$$G^2 a = fa, \text{ and } F^2 = f'a, \text{ and therefore, } \frac{G}{F} = \sqrt{\frac{fa}{af'a}}.$$

Now since the proposed orbit is nearly circular, $(\text{vel.})^2$ at apsidal distance (a) = force $\times$ a nearly, $= \frac{1}{a^2} . fa$, and since at an apse the velocity is wholly transverse, $(\text{vel.})^2$ at V in orbit $Vp = \frac{G^2}{F^2} . (\text{vel.})^2$ at V in orbit VP, $= \frac{G^2}{F^2} . \frac{F^2}{a} = \frac{G^2}{a} = (\text{vel.})^2$ in proposed orbit, since $G^2 a = fa$. Since then in the orbit Vp, and in that of which the apsidal angle is required, the apsidal distances and the forces at equal distances, as well as the velocities at the apsidal distances are equal, the orbits will be similar, and the apsidal angles equal; but the apsidal angle in the orbit Vp

$$= \frac{G}{F} . 180^\circ \text{ (Prop. XLIII. Cor.)} = \sqrt{\frac{fa}{af'a}}\, 180^\circ;$$

and therefore the apsidal angle required $= \sqrt{\frac{fa}{af'a}}\, 180^\circ$.

Ex. 1. Let the force $= \mu r^{n-3}$;

$$\therefore \text{ force } = \frac{\mu}{r^3} r^n = \frac{\mu}{r^3}(a-x)^n$$

$$= \frac{\mu}{r^3}(a^n - na^{n-1} . x), \text{ nearly};$$

$$\therefore fa = \mu a^n, \quad f'a = \mu n a^{n-1};$$

$$\therefore \text{ apsidal angle } = \frac{180^\circ}{\sqrt{n}}.$$

Ex. 2. Let the force $= \frac{\mu r^m + \nu r^n}{r^3}$,

$$\therefore \text{ force } = \frac{1}{r^3}\{\mu(a-x)^m + \nu . (a-x)^n\}$$

$$= \frac{1}{r^3}\{\mu a^m + \nu a^n - (m\mu a^{m-1} + n\nu a^{n-1})x + \&c.\};$$

$$\therefore fa = \mu a^m + \nu a^n,$$

$$f'a = m\mu a^{m-1} + n\nu a^{n-1};$$

$$\therefore \text{ apsidal angle } = \sqrt{\left\{\frac{\mu a^m + \nu a^n}{m\mu a^m + n\nu a^n}\right\}} . 180^\circ.$$

If $a = 1$, apsidal angle $= \sqrt{\left\{\dfrac{\mu + \nu}{m\mu + n\nu}\right\}}\, 180^0$.

In this manner, as will be shewn in the next section, the motion of the moon's apsides might be found approximately, if the direction of the disturbing force of the sun upon the moon tended wholly to the earth's center; but since this is not the case, their motion cannot be determined by the method here proposed.

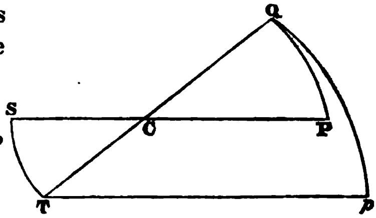
SECTION XI.

ON THE MOTION OF BODIES MUTUALLY ATTRACTING EACH OTHER.

THE motion of a physical point, attracted to an immovable center of force, has been explained in the preceding sections. We now proceed to consider the motions of mutually attracting bodies, of which the masses bear a finite ratio to each other. In this case the attracting body placed in the center of force is no longer immovable, for by the third law of motion the actions of the attracting and attracted bodies are mutual and equal; so that if M represent the mutual attraction of two bodies, whose masses are S and P, the bodies themselves will be acted on by accelerating forces $\frac{M}{S}$ and $\frac{M}{P}$ respectively, and a motion will consequently be generated in each, the nature of which it is now proposed to investigate.

PROP. LVII. *Two bodies attracting each other describe similar figures about their center of gravity, and about each other.*

Let S and P be the bodies, join SP and take $SC : SP = P : S + P$, then C is their center of gravity. If C be in motion, let a motion always equal and opposite to that of C be applied to the system, then C will continue at rest; and since the same motion applied to all the parts of a system produces no alteration in their relative motions, the relative orbits described by S and P about C and about each other will not be affected.



Let ST and PQ be arcs described in the same time round C; then

$$TC : CQ = P : S = SC : CP,$$

$$\therefore\ TC : SC = CQ : CP,$$

and angle SCT = angle PCQ; therefore ST and PQ are similar figures, and they are the figures described about the center of gravity.

Again, draw Tp parallel and equal to Sp. To a spectator at S, who is insensible of his own motion and refers the whole motion to P, P at first will be seen in the direction SCP or Tp, and afterwards in the direction TQ, and will therefore appear to have described the angle pTQ about S,

$$\text{and } SP \text{ or } Tp : CP = S + P : S = TQ : CQ;$$

$$\therefore\ Tp : TQ = CP : CQ,$$

and angle pTQ = angle PCQ, therefore the curves pQ and PQ are similar, that is, the figure described by P round S in motion is similar to the figures described by P and S round their center of gravity.

Prop. LVIII. *An orbit similar and equal to the apparent orbit of* P *round* S *in motion may be described round* S *fixed by the action of the same central force.*

Let PQ and ST be the similar orbits described by P and S round C, their center of gravity. Take

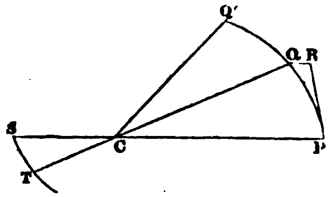
$$Sp = SP, \quad \angle\, pSq = \angle\, PCQ,$$

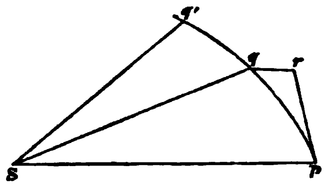
and take Sq such, that

$$Sq : Sp = CQ : CP = TQ : SP;$$

$$\therefore\ Sq = TQ,$$

and therefore q traces out the apparent orbit of P. Draw the subtenses QR, qr, parallel to CP, Sp, and meeting the tangents at P, p in R, r.





Let a body be projected from p with a velocity v, which is to V the velocity at P,

$$\text{as } \sqrt{S+P} : \sqrt{S}, \quad \text{as } \sqrt{Sp} : \sqrt{CP}, \quad \text{as } \sqrt{pr} : \sqrt{PR}$$

by similar figures, and let T, t be the times of describing PR, pr; then ultimately

$$\frac{T}{t} = \frac{PR}{V} \div \frac{pr}{v} = \frac{PR}{pr} \cdot \sqrt{\frac{pr}{PR}} = \sqrt{\frac{PR}{pr}} = \sqrt{\frac{QR}{qr}}.$$

Also the force being the same,

$$\frac{\text{space through which } P \text{ is drawn in } T''}{\text{space through which } p \text{ is drawn in } t''} = \frac{T^2}{t^2} = \frac{QR}{qr} \text{ ultimately},$$

but RQ = space through which P is drawn in T'',

$$\therefore\ rq = \text{.......................} \ p \ \text{............} \ t'';$$

and therefore q is the place of the body at the end of t''; it will also continue in the curve, for the forces being equal and the orbits similar, the resolved parts of the forces in the directions of the tangents will be equal at all corresponding points in the arcs PQ, pq; hence the increments of the velocities continually generated, as the bodies describe the arcs, will be ultimately as the times of describing similar arcs, that is,

$$\text{as } T : t, \text{ as } \sqrt{S} : \sqrt{S+P};$$

$$\therefore \text{ componendo, vel. at } q : \text{vel. at } Q = \sqrt{S+P} : \sqrt{S},$$

hence the body is under the same circumstances as at p, and will therefore continue in the curve.

COR. 1. *Two bodies, which attract each other with forces varying as the distance, describe similar ellipses about their center of gravity and about each other as centers.*

For the orbits described about C and about each other are similar to that described about S fixed, which in this case is an ellipse, whose center is S.

COR. 2. *Two bodies, which attract each other with forces varying inversely as the square of the distance, describe similar ellipses about their center of gravity and about each other as foci.*

COR. 3. *Two bodies revolving round their center of gravity describe round it areas proportional to the times.*

Let PQ, PQ' be arcs respectively similar to pq, pq', and let T, T', t, t', be the times of describing the four arcs respectively;

$$\text{now } \frac{t}{T} = \frac{\sqrt{S+P}}{\sqrt{S}} = \frac{t'}{T'};$$

$$\therefore \frac{t}{t'} = \frac{T}{T'};$$

also by similar figures,

$$\frac{\text{area } PCQ}{\text{area } PCQ'} = \frac{\text{area } pSq}{\text{area } pSq'} = \frac{t}{t'} = \frac{T}{T'};$$

$$\therefore \text{ area } PCQ \propto \text{time of describing it.}$$

Prop. LIX. *The periodic time of* P *round* S *at rest* : *that of* P *or* S *round* C = $\sqrt{\text{S}+\text{P}} : \sqrt{\text{S}}$.

For the orbits, being similar, may be divided into the same number of similar parts, as pq, PQ in Prop. LVIII;

$$\text{and time of describing } pq$$
$$: \text{time of describing } PQ = \sqrt{S+P} : \sqrt{S},$$

and the same being true for the times of describing all the similar arcs, we have componendo

$$\text{periodic time of } P \text{ round } S \text{ at rest}$$
$$: \text{that of } P \text{ or } S \text{ round } C = \sqrt{S+P} : \sqrt{S}.$$

Prop. LX. *Force* $\propto (dist)^{-2}$. *If* (a) *be the axis major of the apparent orbit described by* P *round* S *in motion,* (a') *that of an orbit described by* P *round* S *at rest in the same periodic time, then* $a : a' = \sqrt[3]{\text{S}+\text{P}} : \sqrt[3]{\text{S}}$.

Let $p'q'$ be the ellipse, of which a' is the axis major, that is, let $p'q'$ be an ellipse described by P round S at rest, in the same periodic time as that in which P describes an ellipse round S in motion, or as that in which PQ is described; and let pq be the apparent orbit described by P round S in motion; then, since the force in the two orbits is the same,

$$\text{periodic time in } p'q' : \text{periodic time in } pq = a'^{\frac{3}{2}} : a^{\frac{3}{2}} \text{ (Prop. xv.)}$$

also by Prop. LIX,

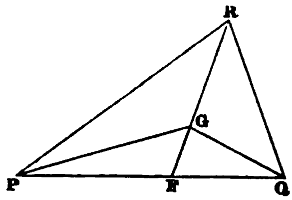
$$\text{period. time in } pq : \text{period. time in } PQ = \sqrt{S+P} : \sqrt{S}.$$
$$\therefore \text{period. time in } p'q' : \text{period. time in } PQ = \sqrt{(S+P)a'^3} : \sqrt{Sa^3},$$

the first term of which proportion is equal to the second by the hypothesis,

$$\therefore (S+P)\, a'^3 = Sa^3;$$
$$\therefore a : a' = \sqrt[3]{S+P} : \sqrt[3]{S}.$$

Prop. LXIV. *To determine the motion of a system of bodies attracting each other with forces varying as the distance between their centers.*

Let P and Q be two bodies collected in their respective centers of gravity. Join PQ and take $PF : PQ = Q : P + Q$, then F is the center of gravity of P and Q; and $(P + Q) . PF = Q . PQ =$ force of Q on P; but $(P + Q)\ PF =$ the force, which two bodies equal to P and Q placed at F would exert on P, therefore P is attracted in the same manner as if a body equal to the sum of the bodies were placed at F, and will therefore describe an ellipse round F at rest as its center. Similarly Q will describe an ellipse round the same point as a center, and in the same periodic time, since the absolute force $P + Q$ is the same in both cases.



Let R be a third body, join RP, RQ, RF; the forces $R . PR$ and $R . QR$, which R exerts on P and Q, may be resolved respectively into $R . PF$, $R . FR$, and $R . QF$, $R . FR$; the force $R . FR$, being the same for either body, produces no disturbance in their relative motions, and therefore the bodies will move in the same manner with respect to each other, as if that force did not act. The other forces $R . PF$, $R . QF$, varying as the distance of P and Q from F, will not cause any perturbations in the orbits described by P and Q round F, and therefore these bodies will still describe ellipses round F, but since the absolute force is increased in the ratio of $P + Q + R : P + Q$, the periodic time will be diminished in the ratio of $\sqrt{P + Q} : \sqrt{P + Q + R}$.

Again, in FR take $FG : FR = R : P + Q + R$, and join PG, QG; then G is the center of gravity of P, Q, R; and $R . FR = (P + Q + R) . FG$; hence the force which R exerts on P is equivalent to the forces $R . PF$ and $(P + Q + R) . FG$, and the force which Q exerts on P is equal to $(P + Q) . PF$; hence the whole force on P is equal to $(P + Q + R)\ PF$ and $(P + Q + R)\ FG$, that is, to $(P + Q + R)$

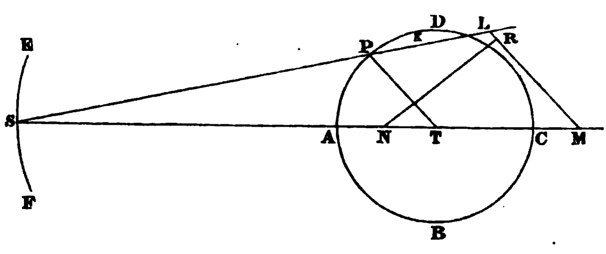
PG, and therefore *P* will describe an ellipse round *G* as a center. Similarly *Q* will describe an ellipse round the same point as a center, and therefore *P* and *Q* describe ellipses round their common center of gravity and round the center of gravity of the system.

In the same manner it may be shewn that *P* and *R*, and *Q* and *R* will describe ellipses round their common centers of gravity respectively, and round the center of gravity of the system; and the same may be proved of any number of bodies.

PROP. LXVI. *Force* $\propto$ *(dist.)*$^{-2}$. *Two bodies* S *and* P *revolve round a third* T *in such a manner, that* P *describes the interior orbit: to shew that* P *will describe round* T *areas more nearly proportional to the times, and a figure more nearly resembling an ellipse, if* T *be acted on by the attractions of the other two, than if it were either not attracted by them at all, or attracted much more or much less.*

Let *PAB*, *ESF* be the orbits of *P* and *S* respectively.

1. Let the orbits be in the same plane. Join *SP*, *PT*, *TS*, and in *SP*, produced if necessary, take *KS* equal to the mean distance of *P* from *S*, and let it represent the accele-



rating force of attraction of *P* to *S* at that distance; take also $LS = \frac{KS^2}{PS^2} \cdot KS$, then *LS* will represent the attraction of *P* to *S* at the distance *PS*. Draw *LM* parallel to *PT* meeting *ST*, produced if necessary, in *M*, and resolve *LS* into the forces *LM*, *MS*.

R.

P is acted on by three forces, LM, MS and its original gravitation to T, the last of which would cause it to describe areas proportional to the times and an ellipse, focus T: the force LM, acting in the direction PT, does not affect the equable description of areas, but since by composition with the attraction of T on P it forms a force not varying as $(\text{dist.})^{-2}$, it will disturb the elliptic form of P's orbit; and the force MS, neither acting in the direction PT, nor varying as $(\text{dist.})^{-2}$, will disturb both the equable description of areas and the elliptic form of the orbit.

Let NS represent the attraction of S on T; then if MS and NS are equal, these equal forces, acting in parallel directions on P and T, will not disturb the relative motions of the two bodies; but if they are unequal, the disturbing force on P will be represented by their difference MN; hence the less MN is, the smaller will be the disturbances produced: now since the distance of P from S is sometimes greater and sometimes less than that of T from S, the mean attraction KS of P to S differs less from NS, than if T were attracted by a *much* greater or *much* less force; that is, the disturbing force MN will be less, and therefore the equable description of areas and the elliptic form of P's orbit will be less disturbed, if T be attracted by S, than if it were not attracted by S at all, or attracted much more or much less.

DEF. The *Line of Nodes* is the straight line, in which the planes of the orbits of P and S intersect each other.

2. Let the orbits lie in different planes. The same construction being made, the force LM acting in direction PT, which is in the plane of P's orbit, produces the same effect as in the first case, and has no tendency to draw P from the plane of its orbit. But MN, acting in a direction inclined to that plane, except when the line of nodes passes through S, not only produces the effects mentioned in the first case, but also tends to draw P from the plane of its orbit; and this and the other perturbations depending on the magnitude of MN will be least, when MN is least, that is when NS is equal or nearly equal to KS, as before

Obs. In the proposition P is supposed to describe an orbit round T fixed; this cannot in reality be the case, as long as its magnitude bears a finite ratio to that of T; for, leaving out the consideration of the forces which S exerts, the two bodies P and T describe orbits about their center of gravity. The orbit here meant is the *apparent* orbit of P to a spectator at T, that is, the orbit $p\,Q$ in Prop. 57. If, however, we suppose a force applied every instant to P and T equal and opposite to that which P exerts on T, T will remain at rest, and the gravitation of P to T will be the sum of the attractions of T on P, and of P on T, acting in the direction PT; so that the whole gravitation of P to $T = \dfrac{P+T}{PT^2}$.

Prob. I. *To investigate expressions for the disturbing forces of* S *on* P, *on the supposition that* P's *orbit is circular, and coincident with the plane of* S's *orbit.*

Force of S on P represented by $LS = \dfrac{S}{SP^2}$,

$\therefore$ force of S on P in direction PT

$$= \frac{S}{SP^2} \cdot \frac{LM}{LS} = \frac{S}{SP^2} \cdot \frac{PT}{SP} = \frac{S \,.\, PT}{SP^3} \quad \dots\dots\dots (1),$$

this is called the *addititious* force, and is represented by LM.

Again, force of S on P in direction TS

$$= \frac{S}{SP^2} \cdot \frac{MS}{LS} = \frac{S}{SP^2} \cdot \frac{SP}{ST} = \frac{S \,.\, ST}{SP^3},$$

and force of S on T in direction $TS = \dfrac{S}{ST^2}$;

$\therefore$ disturbing force of S on P in direction TS

$$= S \left\{ \frac{ST}{SP^3} - \frac{1}{ST^2} \right\} \dots\dots\dots\dots\dots\dots (2),$$

this is called the *ablatitious* force, and is represented by MN.

Draw NR perpendicular to LM; then MN is equivalent to MR, RN,

$$RN = MN \sin NMR = S.\left\{\frac{ST}{SP^3} - \frac{1}{ST^2}\right\} \sin PTS, \ldots (3),$$

this force acts in the direction of the tangent at P, and is called the *tangential* force.

$$\text{Similarly, } MR = S.\left\{\frac{ST}{SP^3} - \frac{1}{ST^2}\right\} \cos PTS,$$

$$\text{hence } LR = LM - MR = \frac{S.PT}{SP^3} - S.\left\{\frac{ST}{SP^3} - \frac{1}{ST^2}\right\} \cos PTS, \ldots (4),$$

this force, which is the resultant of the disturbing forces of S on P in direction PT, is called the *central disturbing* force.

Hence the gravitation of P to T

$$= \frac{P+T}{PT^2} + S.\left\{\frac{PT}{SP^3} - \left(\frac{ST}{SP^3} - \frac{1}{ST^2}\right) \cos PTS\right\}.$$

Prob. II. *To find approximate expressions for the above disturbing forces, when* ST *is very great compared with* PT.

$$SP = \{ST^2 - 2ST.PT \cos PTS + PT^2\}^{\frac{1}{2}}$$

$$= ST\left\{1 - \frac{2PT}{ST} \cos PTS\right\}^{\frac{1}{2}} \text{ nearly};$$

$$\therefore \frac{1}{SP^3} = \frac{1}{ST^3}\left\{1 + \frac{3PT}{ST} \cos PTS\right\} \text{ nearly},$$

$$\therefore \frac{ST}{SP^3} - \frac{1}{ST^2} = \frac{3PT}{ST^3} \cos PTS,$$

hence the ablatitious force

$$= \frac{3S.PT}{ST^3} \cos PTS,$$

the tangential force

$$= \frac{3S.PT}{ST^3} \cos PTS . \sin PTS = \frac{3S.PT}{2ST^3} \sin 2PTS,$$

the central disturbing force

$$= \frac{S.PT}{ST^3}\left\{1 + \frac{3PT}{ST}\cos PTS\right\} - \frac{3S.PT}{ST^3}\cos^2 PTS$$

$$= \frac{S.PT}{ST^3}.\left\{1 - 3\cos^2 PTS\right\} \text{ nearly}$$

$$= -\frac{S.PT}{2ST^3}\left\{1 + 3\cos 2PTS\right\}.$$

Cor. 1. Let F = the mean central disturbing force, or the force, which, acting uniformly for a whole revolution of P round T, would produce the same effect as the variable central disturbing force; and let the four right angles through which TP moves in one revolution be divided into n equal angles; then

$$F = -\frac{S.PT}{2ST^3}.\frac{1}{n}\left\{n + 3\left(\cos\frac{4\pi}{n} + \cos\frac{8\pi}{n} + \cos\frac{12\pi}{n} + \dots \right.\right.$$

$$\left.\left. + \cos\frac{4n\pi}{n}\right)\right\} \text{when } n \text{ is infinite,}$$

$$= -\frac{S.PT}{2ST^3}\left\{1 + \frac{3}{n}.\frac{\cos\left(\frac{n+1}{n}.2\pi\right)\sin 2\pi}{\sin\frac{2\pi}{n}}\right\} \text{when } n \text{ is infinite,}$$

$$= -\frac{S.PT}{2ST^3},$$

and therefore the mean central disturbing force is ablatitious, and diminishes the gravitation of P to T.

Def. 1. P is said to be in *syzygy*, when its orthogonal projection on the plane of S's orbit lies either in ST or in ST produced, and in *quadrature* when the projections lie in a line drawn through T in the plane of S's orbit perpendicular to ST.

In the first nine corollaries to the Proposition the planes of the two orbits are supposed to coincide, and therefore P will be in syzygies at A and C, when crossing the line ST or ST produced, and in quadratures at B and D, 90° distant from A or C. S and P move in the directions ESF,

The lunar variation arises from the tan-
gential disturbing force, which affects the
place $= 35' 42'' \sin 2(☽ - ☉)$

☽ moon's longitude

☉ sun's long.

5. If the moon's orbit had been originally a circle the effect of the tangential disturbing force would have made it an oval, with major axis in quadrature, because where her velocity is quickest her orbit is flattest.

6. Annual $=$ " $= 11' 12'' \sin m$

$m =$ ☉'s mean anomaly

DAB. The distance *PS* is supposed invariable, and so great as to be always considered parallel to *TS*. In the eighth and ninth corollaries the eccentricity of *P*'s orbit is taken into account, but the expressions above obtained for the disturbing forces on the supposition that *P*'s orbit is circular, may, on account of the smallness of the eccentricity, be applied without affecting the *general* correctness of the results deduced.

Cor. 2. If the planes of the two orbits coincide, the central disturbing force $= -\dfrac{2S.PT}{ST^3}$ when *P* is in syzygies, and $=\dfrac{S.PT}{ST^3}$ when *P* is in quadratures; and is therefore ablatitious in the former case, and addititious in the latter.

Def. 2. If the Earth, Moon and Sun be supposed to be represented by *T*, *P*, and *S*, the Moon is said to be in *perigee* when at the nearer, and in *apogee* when at the farther apse.

Corollaries to the Proposition.

Cor. 1. What has been proved as to the disturbances caused by *S* may be proved as to those produced by any other body revolving round *T*: hence if several bodies *P*, *S*, *R*, &c. revolve about another *T*, the motion of the innermost body *P* will be least disturbed by the attractions of *S*, *R*, &c. when *T* is attracted by the others in the same manner as they mutually attract each other.

Cor. 2. *The areas, described by* P *round* T *in the same given times, continually increase as* P *moves from quadrature to syzygy, and continually decrease from syzygy to quadrature.*

For the only part or the disturbing force, which affects the equable description of areas is the tangential force, and it acts in consequentiâ from upper quadrature to syzygy, and in antecedentiâ from syzygy to lower quadrature.

Similarly the areas described in the same given times increase continually from lower quadrature to syzygy, and decrease from syzygy to upper quadrature.

Cor. 3. *The velocity of* P *is greatest in syzygies, and least in quadratures.*

Cor. 4. *If* P's *orbit be originally circular, the curvature of the disturbed orbit will be greatest in quadratures, and least in syzygies.*

For the radius of curvature in an orbit nearly circular $\propto \frac{(\text{vel})^2}{\text{central force}}$, and therefore the curvature, which varies inversely as the radius varies as $\frac{\text{force}}{(\text{vel})^2}$. Now the force of P to T is greatest in quadratures, and least in syzygies, and the velocity of P is least in the former case, and greatest in the latter; hence on both accounts the curvature is greatest in quadratures and least in syzygies.

Cor. 5. *Hence* P's *orbit, if it be originally circular, will assume the form of an oval, whose axis major passes through quadratures, and axis minor through syzygies.*

Cor. 6. *To consider the effect produced by the disturbing forces on the periodic time of* P *round* T.

The tangential force accelerates and retards P's motion equally in a whole revolution, and therefore does not affect the periodic time. But the central disturbing force in a whole revolution diminishes the gravitation of P to T, and therefore increases the distance PT; hence the periodic time, which $\propto \frac{(\text{rad})}{\sqrt{\text{absolute force}}}$, will from both these causes be increased by the action of the central disturbing force.

Obs. If S approach towards the system T and P, the central disturbing force, which varies inversely as ST^3, will be increased, and consequently the gravitation of P to T will be still more diminished, and the distance PT increased; hence the periodic time will be still farther increased.

Cor. 7. *The orbit of* P *being supposed nearly circular, to consider the effect of the central disturbing force on the motion of its apsides during a whole revolution.*

Let $PT = r$, and let $\frac{\mu}{r^2}$ represent the force of T on P; then if νr represent the addititious force, when P is in quadrature, $-2\nu r$ will represent the ablatitious force when P is in syzygy; and therefore the whole attractions of P to T in quadrature and syzygy respectively will be $\frac{\mu}{r^2} + \nu r$, and $\frac{\mu}{r^2} - 2\nu r$. Hence if the force in quadratures prevailed for a whole revolution, the apsidal angle would $= \sqrt{\frac{\mu + \nu}{\mu + 4\nu}} \cdot 360^0$. which is less than 360^0, or the apside would regrede; and if the force in syzygies prevailed for the same time, it would $= \sqrt{\frac{\mu - 2\nu}{\mu - 8\nu}} \cdot 360^0$, which is greater than 360^0, or the apside would progrede. At any other point the apside will regrede or progrede, according as the disturbing force at that point increases or diminishes the gravitation of P to T; but the gravitation is on the whole diminished by the central disturbing force, and therefore its tendency is to make the apsides progrede.

Obs. In investigating in this and the following Corollaries the effects produced on P's orbit by the different disturbing forces, it is to be observed that only general results are obtained: the disturbing force may be supposed to act by impulses, its effects are then examined at the points where its action is most effective, and from these a general conclusion is drawn as to its effect in a whole revolution of P.

Cor. 8. *The orbit to* P *being supposed eccentric, to consider the effect of the central disturbing force on the motion of its apsides.*

1. Let the apsidal line be in a syzygy; draw the tangent Py in the direction of P's motion. As P approaches perigee,

the

forc
rour
dist
TP
then

$F =$

=

=

and
anc

prc
ST
in
cul

of
wil
or
tan

the central disturbing force being ablatitious*, tends to draw P from T; hence the *acute* angle TPy is increased by it, or P arrives at an apse (π) sooner than it would have done in the undisturbed orbit; therefore the apsidal line regredes. For a short time after passing perigee, the disturbing force, being still ablatitious, tends to increase the *obtuse* angle TPy, so that P appears to have proceeded from an apse (π') still more distant than π; hence if the disturbing force now ceased acting, so that P described an undisturbed ellipse, the apogee, found by producing $\pi' T$, will have regreded more than that found by producing πT, and therefore both before and after perigee, the tendency of the central disturbing force is to make the apsidal line regrede. As P approaches near to apogee, the disturbing force being still ablatitious increases the obtuse angle TPy, and $\therefore$ P arrives at the apse later than it would otherwise have done, or the line of apses progredes; and in like manner as before it may be shewn to progrede still farther after P leaves apogee; hence when P is near apogee the line of apses is progressive. Now the disturbing force, varying as PT, is greater in the latter case than in the former, hence the progression of the apsidal line, when P is near apogee, is greater than the regression, when P is near perigee. When P is near the extremities of the latus rectum, it may be easily seen by reasoning similar to the above, that the effect of the disturbing force is to make the apsidal line regrede at one extremity, and progrede at the other, and PT being in this case the same for both, the regression will equal the progression. Similarly, at other intermediate points between syzygies and quadratures, the disturbing forces in a whole revolution will in a great measure counteract each other, and their effects need not be considered; hence when the apsidal line is in syzygy, the effect of the central disturbing force is to make it progrede.

2. Let the apsidal line be in quadrature: then at the apsides the disturbing force is addititious; and it may be shewn as above, that when P is near perigee, the apsidal line

---

* In this and the remaining Corollaries, the central disturbing force is called ablatitious, when it acts in the direction TP, and therefore tends to diminish the gravitation of P to T.

Lunar evection $= 1^{\circ}\,20'\,30'' \times \sin 2\{(☽ - \odot) - M\}$

$M =$ ☽'s mean anomaly.

progredes, and regredes when P is near apogee; and the regression in this case is greater than the progression; therefore since the whole motion of the apsides for other positions of P is inconsiderable, in this position the apsidal line is regressive.

The apsidal line then progredes when in syzygy, and regredes in quadrature; but the progression exceeds the regression; for the former is due to the difference of the ablatitious forces at apogee and perigee, when the apsidal line is in syzygy, and the latter to the difference of the addititious forces at the same point, when that line is in quadrature, and the former difference equals twice the latter. As the line of apses by the actual motion of S appears to revolve from syzygy to quadrature, the progression for the same reason exceeds the regression; hence during a whole revolution of S the effect of the central disturbing force is to make the line of apses progrede.

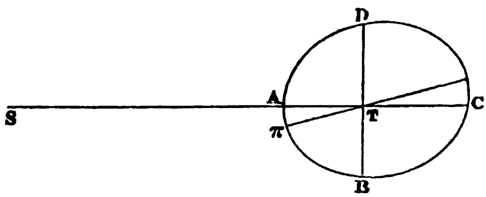
Moreover, when the apsidal line is in syzygy and therefore progressive, it is moving in the same direction as S, and thus continues longer in syzygy than if S were quiescent, and hence the progression is increased. When the apsidal line is in quadrature, the contrary takes place, and the regression is not so great as if S were stationary. (Vid. Prof. Airy's "Gravitation.")

Cor. 9. *To consider the effect of the central disturbing force on the eccentricity of* P's *orbit.*

As P moves from perigee to apogee, the ablatitious force tends to increase, and the addititious force tends to diminish the *obtuse* angle TPy, which the tangent Py makes with PT; also the velocity at any point, and therefore the axis major remains nearly unaltered; therefore in the former case the form of the orbit departs farther from, and in the latter approaches nearer to that of a circle; that is, the tendency of the ablatitious force is to increase, and that of the addititious to diminish the eccentricity. As P moves from apogee to perigee, the *acute* angle TPy is increased by the former force, and diminished by the latter, that is, the eccentricity is diminished by the ablatitious and increased by the addititious force.

1. When the line of apsides is in either syzygy or quadrature, the effects in either case of these disturbing forces separately, as P moves from perigee to apogee, are equal and opposite to those produced by them during P's motion from apogee to perigee; and therefore the eccentricity of P's orbit in either of these positions of the apsidal line is unaltered by the central disturbing force.

2. Let the perigee π lie between lower quadrature and nearer syzygy.



At A and C the disturbing force is ablatitious, and at the former point P is moving towards, and at the latter from perigee; hence at A the force tends to diminish, and at C to increase the eccentricity; but TC is greater than TA, and $2 \times$ distance is a measure of the ablatitious force at these points, therefore the combined effects of the forces at A and C will increase the eccentricity. At B and D the force is addititious, and at B, P is moving from, and at D towards perigee, hence the tendency of the force at B is to diminish, and at D to increase the eccentricity; but TD is greater than TB, and the distance is a measure of the addititious force at these points, therefore on the whole the forces at B and D increase the eccentricity. Hence in this position of the apsidal line the eccentricity is increased in a whole revolution.

Now as S moves in a direction parallel to AB, π moves from quadrature towards syzygy, and therefore $2(TC - TA)$ continually increases, and $TD - TB$ decreases, but the former difference increases faster than the latter decreases; hence, as the perigee moves from quadrature to syzygy, the eccentricity is continually increasing.

3. By reasoning similar to the above it may be shewn, that as the perigee moves from syzygy to upper quadrature,

the eccentricity is continually decreasing; that it increases as it moves through DC, and decreases through CB; so that generally, *the eccentricity continually increases as the apsidal line revolves from quadrature to syzygy, and decreases as that line revolves from syzygy to quadrature.*

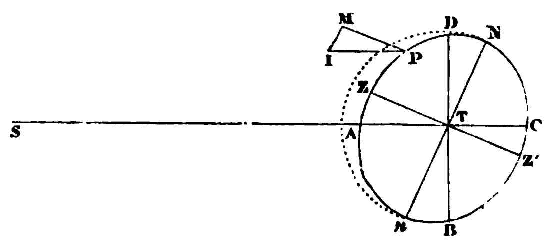
COR. 10. *To consider the effects produced on the inclination of* P's *orbit to that of* S *by the ablatitious force.*

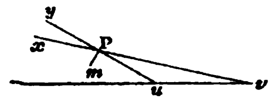
Let Nn be the line of nodes: through P draw PI parallel to TS to represent the ablatitious force at P, IM perpendicular to the plane of P's orbit, and join PM: the force PI may be resolved into the two PM, MI, of which the latter only affects the inclination of the orbit; and since during P's motion from upper to lower quadrature the ablatitious force acts in direction TS or PI, and through the remaining part of the orbit in direction ST or IP, the perpendicular force in the former case acts in direction MI, and in the latter in direction IM; hence the perpendicular force tends towards the plane of S's orbit through Dn and BN, and from it through nB and ND; and similarly, whatever be the position of the nodal line, the perpendicular force tends towards the plane of S's orbit, except when P is between quadrature and the nearer node.

In the plane of P's orbit, draw ZTZ' perpendicular to Nn.

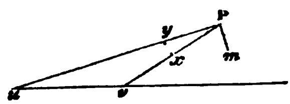
1. When the nodes are in syzygy, since no part of the disturbing force acts out of the plane of P's orbit, the inclination will not be affected by it.

2. Let the node N lie between upper quadrature and farther syzygy, and let the portion NPn of its orbit be above the plane of that of S. From upper quadrature to Z, P is

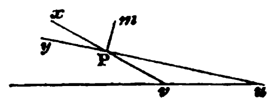




moving from the plane of S's orbit; let Py the tangent at P (fig. 2.) be produced backward to meet that plane in u, draw Pm parallel to MI; then Px the new direction of P's motion will fall between Py and Pm, and when produced backwards will cut the plane in v at a less angle than that in which yP cuts it, and therefore the inclination of P's orbit, the position of which is determined by the point T and the direction of P's motion, is diminished. From Z to n, P is moving *towards* the plane of S's orbit, and therefore, as appears from fig. 3, Px will cut the plane at a greater angle than that in which Py cuts it, or the inclination is increased.



From n to lower quadrature P is moving from the plane, and the perpendicular force now tends from the plane, and therefore, as in fig. 4, the inclination is increased.



In a similar manner it may be shewn, that as P moves from B to Z' the inclination is diminished, that it increases from Z' to N, and also from N to D; hence if $NTD = a$, the inclination in this position of the line of nodes is increased, while P describes $180^0 + 2a^0$ and diminished through $180^0 - 2a^0$.

3. When the nodes are in quadrature the inclination is as much increased as it is diminished, and therefore at the end of one revolution it is unaffected by the ablatitious force.

4. Let N lie between C and B at an angular distance (a) from B; then it may be shewn by reasoning similar to the above, that in this position the inclination is increased, while P moves through $180^0 - 2a^0$, and diminished through $180^0 + 2a^0$.

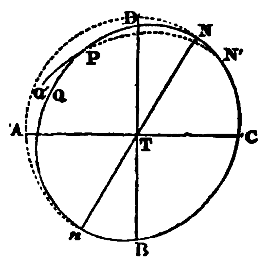
As the node recedes (see Cor. 11.) from quadrature to syzygy, the inclination is increased, and from syzygy to quad-

rature it is as much diminished, so that in a whole revolution of the nodes the inclination is neither increased nor diminished. The inclination is a maximum when the nodes are in syzygy, and a minimum when they are in quadrature; and least of all when the nodes are in quadrature and P in syzygy.

Cor. 11. *To consider the effects produced on the motion of the Nodes by the ablatitious force.*

Let P be the place of the body; resolve the ablatitious force at P into two, one perpendicular to and the other in the plane of P's orbit; and let PQ be a small arc of the orbit which P would describe, were there no perpendicular force; PQ' a small arc of the disturbed orbit.

Then it is manifest that when P is ascending from the node, N' the node of PQ' will lie behind or before N, that is, the node will be retrograde or progressive, according as Q' is at a less or greater distance from the plane of S's orbit than Q, that is, according as the perpendicular force tends towards or from that plane; and the same is true of the node n, when P is approaching that node. Now by what has been shewn in the first part of Cor. 10, the force tends always towards the plane, except between quadrature and the nearer node; hence the motion of the node is always retrograde, except when P is moving between quadrature and the nearer node.



If a be the angular distance of the node from quadrature, the node will be progressive while P moves through $2a^0$, and retrograde through $360^0 - 2a^0$.

Since a is less than 90 except at syzygy, the nodes in a whole revolution of P regrede more than they progrede.

If the nodes be in quadratures, they will regrede during the whole revolution; when they are in syzygies, the disturbing force acting in the plane of P's orbit, produces no

effect upon the node, which therefore remains stationary; it will however pass out of syzygy by the motion of S, and become retrograde.

Cor. 12. *The effects produced by the disturbing forces are greater, when* P *is in conjunction than when in opposition.*

For when P is at nearer syzygy or in conjunction, the addititious force $= \dfrac{S \cdot PT}{SA^3}$, and when at farther syzygy or in opposition, it $= \dfrac{S \cdot PT}{SC^3}$; and SA being less than SC, the former value is greater than the latter. Also in the former case the ablatitious force $= \dfrac{3S \cdot PT}{SA^3}$, and in the latter it $= \dfrac{3S \cdot PT}{SC^3}$, and therefore is greater in conjunction than in opposition. Hence, the effects produced by these forces will be greater in conjunction than in opposition.

Cor. 13. The reasoning employed in this proposition is wholly independent of the magnitude of S; if therefore S be so great, that the system of P and T revolves round S fixed, the disturbing forces will be of the same *kind* as when S moved round T fixed; but since each varies as S, they will all be increased in the same ratio as that in which we suppose S to be increased.

Cor. 14. *If* S *and the distance* ST *vary, whilst the system of* P *and* T *remains the same, the angular error of* P *as seen from* T, *produced in a given time by the disturbing force of* S, *will vary inversely as the square of the periodic time of* T *round* S, *or directly as the cube of the apparent diameter of* S *as seen from* T.

For let S' and $S'T$ be other values of S and ST; then in any given position of P, since PT is the same, the disturbing forces of S on P are to those of S' as $\dfrac{S}{ST^3} : \dfrac{S'}{S'T^3}$,

and therefore the linear errors produced by them in the same unit of time are in the same ratio, and PT being given, the angular errors as seen from T will be proportional to the linear errors; and the same being true of all corresponding angular errors, componendo, the angular errors generated in a given time will be as $\frac{S}{ST^3} : \frac{S'}{S'T^3}$, that is, by Prop. xv

$$\text{as } \frac{1}{(\text{per}^{\text{c}}.\ \text{time})^2 \text{ of } T \text{ round } S} : \frac{1}{(\text{per}^{\text{c}}.\ \text{time})^2 \text{ of } T \text{ round } S'},$$

and therefore the angular error varies inversely as

$$(\text{per}^{\text{c}}.\ \text{time})^2 \text{ of } T \text{ round } S.$$

Also if D = diameter of S, $S \propto D^3$, and therefore angular error $\propto \frac{D^3}{ST^3} \propto$ the cube of the apparent diameter of S, as seen from T.

Cor. 15. *If there be two systems* P, T, S *and* P′, T′, S′, *such that* S : S′ = T : T′, *and* PT : ST = P′T′ : S′T′; *and if the orbits of* P *and* P′ *be similar and similarly situated, their periodic angular errors round* T *and* T′ *arising from the disturbing forces of* S *and* S′ *will be equal.*

The bodies P and P' at any two similarly situated points in each orbit, are similarly acted on by proportional forces, and therefore the linear errors, generated while they move through small similar parts of their orbits, will be similar and proportional, and will therefore be respectively as the diameters of the orbits; hence, the angular errors through those small parts will be equal; and this being true of the errors through all corresponding parts, the periodic angular errors will be equal.

Cor. 16. *In any two systems* P, T, S *and* P′, T′, S′, *in which the orbits of* P *and* P′ *are similar and similarly situated, to compare the periodic angular errors round* T *and* T′.

Let P and p be the periodic times of T round S and of P round T,
P' and p' .... T' ...... S' and of P' ...... T'.

In TS, produced if necessary, place a body s such that $s : S' = T : T'$, and at a distance sT from T, such that

$$sT : PT = S'T' : P'T'; \quad \therefore s = \frac{T}{T'} S', \text{ and } sT = \frac{PT}{P'T'} \cdot S'T'.$$

Then by Cor. 15, the periodic angular errors in the system P', T', S' equal the errors in the system P, T, s. Again, by Cor. 14, in the systems P, T, S and P, T, s, the angular errors in a given time, and therefore the periodic angular errors are

$$\text{as } \frac{S}{ST^3} : \frac{s}{sT^3} \text{ as } \frac{S}{ST^3} : \frac{S'}{S'T'^3} \cdot \frac{T}{T'} \cdot \frac{P'T'^3}{PT^3},$$

$$\text{as } \frac{S}{ST^3} \cdot \frac{PT^3}{T} : \frac{S'}{S'T'^3} \cdot \frac{P'T'^3}{T'} \text{ as } \frac{p^2}{P^2} : \frac{p'^2}{P'^2};$$

therefore the periodic angular errors in the systems P, T, S and P', T', S' are as $\frac{p^2}{P^2} : \frac{p'^2}{P'^2}$.

Hence, if the orbits of two satellites be similar, and equally inclined to the orbits of their primaries, the periodic angular errors in their orbits will vary directly as the squares of the periodic times of the satellites, and inversely as the squares of those of the primaries.

The errors here spoken of are the angular motions of the nodal line, apsidal line, &c.

Cor. 17. *To compare the mean addititious force with the force of* T *on* P.

Let P be the periodic time of T round S.

p that of P and T round their center of gravity;

therefore $\sqrt{\dfrac{P+T}{T}}\,.\,p$ = time in which P would revolve round T at rest at the same distance TP, by Prop. LIX.

$$\text{Now, mean addititious force} = \frac{S\,.\,PT}{ST^3},$$

$$\text{and force of } S \text{ on } T = \frac{S}{ST^2};$$

$\therefore$ mean addititious force : force of S on $T = PT : ST$,

and by Prop. IV,

$$\text{force of } S \text{ on } T : \text{force of } T \text{ on } P = \frac{ST}{P^2} : \frac{PT}{p^2} \cdot \frac{T}{P+T};$$

$$\therefore \text{mean addititious force} : \text{force of } T \text{ on } P = \frac{1}{P^2} : \frac{1}{p^2} \cdot \frac{T}{P+T}.$$

The force of T on P here spoken of is that with which T alone draws P, and this force is to that with which P and T are drawn towards each other as $T : P+T$; hence compounding this with the above proportion, we have

mean addit^s. force : force of P and T towards each other $= \dfrac{1}{P^2} : \dfrac{1}{p^2}$.

THE END.